MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1945 DECEMBER 14

Professor H. H. Plaskett, President, in the Chair

The election by the Council of the following Fellows was duly confirmed:—

Geoffrey Vernon Ball, 20 Marshall Grove, Erdington, Birmingham, 23;
Robert Stuart Devonald, B.Sc., 11 Westville Road, Cardiff, S. Wales;
John Harold Ellison, M.A., A.I.N.A., Woodspeen Grange, near Newbury, Berkshire;
Jesse Leonard Greenstein, Assistant Professor of Astrophysics in the University of Chicago, Yerkes Observatory, Williams Bay, Wis., U.S.A.;
William Albert Hiltner, Assistant Professor of Astrophysics, Yerkes Observatory, Williams Bay, Wis., U.S.A.;
Khateeb Hyder Hussain, B.Sc., H.E.H. the Nizam's Electricity Department, Hyderabad, Deccan, India;
Percy John Edward Maycock, 52 Donald Road, Uplands, Bristol, 3;
George Thomas Smith-Clarke, Shenandoah, Stoneleigh Road, Gibbet Hill, Coventry, Warwickshire; and
Thomas Alexander Thomson, B.Sc., 4 Cromwell Street, Wakari, Dunedin, New Zealand.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Gordon Atwater, Director of the Hayden Planetarium and Curator of Astronomy, American Museum of Natural History, Central Park West at 79th Street, New York, U.S.A. (proposed by C. Fisher);
Frederick Edward Brasch, The Library of Congress, Washington, U.S.A. (proposed by F. R. Moulton);
George B. Burnside, Fairhill, Dullatur, Dumbartonshire (proposed by A. D. Ross);
James Henry John Burtt, 6 Elmbourne Road, London, S.W. 17 (proposed by F. M. Holborn);
Dorothy G. Hawkins, M.A., Hacks Wood, Walton-in-Gordano, near Clevedon, Somerset (proposed by C. R. Burch);

James Samuel Houston, B.E., A.M.I.C.E., G.P.O. Box 29F, Hobart, Tasmania (proposed by J. L. Hull); and

William Henry Julian, B.Sc., 8 Trelawny Road, St. Austell, Cornwall (proposed by C. S. Saxton).

Forty-six presents were announced as having been received since the last ordinary meeting.

Dr. J. G. Porter and Mr. H. F. Finch were appointed honorary auditors of the Treasurer's accounts for 1945.

THE IDENTIFICATION OF THE CORONAL LINES

(*George Darwin Lecture, delivered by Professor Bengt Edlén on* 1945 *October* 12)

The coronal emission lines are superimposed on the continuous radiation of the inner parts of the solar corona, the region of their appearance extending from the top of the chromosphere to a height generally not exceeding 10 minutes of arc (half a million kilometers) from the Sun's limb. The lines are responsible for only a small fraction of the total intensity of the corona.

The coronal line spectrum has been studied by most eclipse expeditions since 1869, when the first line was discovered by Young and Harkness. During the last decade Lyot's ingenious invention of the coronagraph, described by him in the George Darwin Lecture of 1939 *, has greatly improved and extended the observations, especially for the infra-red part of the spectrum. At present, as a net result of all observations, the wave-lengths of some twenty coronal emission lines have been established. Six of these lines are much stronger than the rest. These conspicuous lines are: λ 3388, λ 5303, λ 6374, λ 7892, λ 10747 and λ 10798. None of the coronal lines has been observed in a laboratory light-source and there is no real coincidence with any of the Fraunhofer lines nor with any line in the chromospheric spectrum. The coronal lines were in fact a unique feature of the solar corona until in 1932 a number of them showed up in the spectrum of the recurrent nova RS Ophiuchi. Recently the coronal lines have reappeared in Nova Ophiuchi at the outburst of 1942 and have also been observed in T Pyxidis in 1945.

The explanation of "coronium", which will form the subject of this lecture, has a close resemblance to Bowen's solution of the nebulium problem inasmuch as the lines are in both cases caused by transitions from low, metastable levels in various ions of cosmically abundant elements, transitions normally forbidden and able to appear only in sources of very low density. While, however, the carriers of the strongest nebulium lines are found among the light elements, especially oxygen, nitrogen and neon, the coronium lines are due to heavier elements. Thus, the six outstanding lines just mentioned are all due to iron. Another difference is that most of the coronal transitions occur between levels within the ground term, whereas the nebular transitions take place between levels of different terms. The main difference, however, is caused by the much higher stage of ionization of the coronal atoms, which are shown to have lost from 10 to 15 electrons.

Turning to the details of the identification †, I will first mention that the starting-point was Grotrian's discovery of the coincidence between the level separations within the ground terms of the spectrum *Fe*X and *Fe*XI with the wave-numbers of the coronal lines λ 6374 and λ 7892 respectively. The ground state of *Fe*X is a ^2P term arising from the electron configuration $3s^2 3p^5$. If supposed to be real, the first mentioned coincidence means that the red coronal line is emitted when iron atoms 9 times ionized pass from the metastable level $^2P_{\frac{1}{2}}$ which has an energy of 1·94 electron volts, to the ground level $^2P_{\frac{3}{2}}$. The level separations of *Fe*X and *Fe*XI were obtained directly from the extreme ultra-violet spectrum of these ions. Table I contains the significant data relating to these identifications as well as to two additional identifications with similar transitions in the ions of *Ca*XII and *Ca*XIII.

* *M.N.*, 99, 580, 1939.

† The full description is in *Z. Astrophys.*, 22, 30, 1942; see also *Ap. J.*, 98, 116, 1943.

TABLE I

The Experimental Basis for the Direct Identification of Coronal Lines

Transitions in the Extreme Ultra-Violet	λ and Intensity in the Vacuum Spark	ν cm.$^{-1}$	Level Separations	Corona
*Fe*X $3s^23p^5 - 3s^23p^44s$:				
$^2P_{\frac{3}{2}} - {}^2P_{\frac{1}{2}}$	95·338 (1)	1 048 900	15 714	
$^2P_{\frac{1}{2}} - {}^2P_{\frac{1}{2}}$	96·788 (2)	1 033 186		
$^2P_{\frac{3}{2}} - {}^2P_{\frac{3}{2}}$	96·122 (4)	1 040 345	15 660	
$^2P_{\frac{1}{2}} - {}^2P_{\frac{3}{2}}$	97·591 (0)	1 024 685		
Average	...	...	15 687	15 683
*Fe*XI $3s^23p^4 - 3s^23p^34s$:				
$^3P_2 - {}^3D_2$	87·025 (1+)	1 149 095	12 667	
$^3P_1 - {}^3D_2$	87·995 (0)	1 136 428		
$^3P_2 - {}^3S_1$	89·185 (1+)	1 121 265	12 679	
$^3P_1 - {}^3S_1$	90·205 (1)	1 108 586		
Average	...	...	12 673	12 668
*Ca*XII $2s^22p^5 - 2s2p^6$:				
$^2P_{\frac{3}{2}} - {}^2S_{\frac{1}{2}}$	141·036 (8)	709 039	30 028	30 039
$^2P_{\frac{1}{2}} - {}^2S_{\frac{1}{2}}$	147·273 (6)	679 011		
*Ca*XIII $2s^22p^4 - 2s2p^5$:				
$^3P_2 - {}^3P_2$	161·748 (1d)	618 246	24 464	24 465
$^3P_1 - {}^3P_2$	168·412 (00d)	593 782		

The lines λ 6374 and λ 7892 belong to the six strongest coronal lines previously mentioned. The strength of these lines combined with the absence of the analogous transitions of *Fe*VII, well known from their prominent appearance in Nova Pictoris, indicates that the most abundant ionization stages of iron might be found more likely above than below *Fe*X and *Fe*XI. A search for these higher stages, however, has to be based on extrapolations, as the relevant spectra in the extreme ultra-violet could not be analysed with the sufficient accuracy and completeness to fix the ground level separations. Preliminary extrapolations showed that transitions analogous to those of *Fe*X and *Fe*XI should be looked for within the configurations $3s^23p$, $3s^23p^2$, $3s^23p^4$ and $3s^23p^5$ of the elements of the iron group. Of these elements iron is by far the most abundant, next followed by nickel.

The difference between the two levels of the ^2P term of the configurations $3s^23p$ and $3s^23p^5$ is conveniently extrapolated by means of the regular doublet law. As shown in Table II, the fourth root of the level separation increases more and more linearly along the sequences of the iso-electronic ions starting with *Al*I and *Cl*I. The extrapolation through the *Al*I sequence to *Fe*XIV suggests unambiguously the origin of the green coronal line.

The extrapolation to *Fe*XIV is rather extended and naturally somewhat precarious. As a decisive confirmation of the identification of both the green and the red line, however, coronal lines of medium strength are found at exactly the predicted positions for the corresponding transitions in *Ni*XII and *Ni*XVI. The latter positions are predictable with great precision as soon as the coronal wave-numbers are accepted for iron.

In the configurations $3s^23p^2$ and $3s^23p^4$ several types of transitions are possible, as shown in Fig. 1.

These transitions are not so simply extrapolated as the coupling changes considerably along the sequences. Using the known formulæ for intermediate coupling it is possible, however, to follow a procedure consisting essentially of a transformation of the ^3P-

TABLE II

Comparison of the Ground-Term Splittings in the Iso-electronic Sequences $3s^2 3p$ (*Al*I, *Si*II, . . .) *and* $3s^2 3p^5$ (*Cl*I, *A*II, . . .); $\zeta = \frac{2}{3}\Delta\nu$

Atomic Number Z	Ion	$^2P_{\frac{3}{2}} - {}^2P_{\frac{1}{2}}$ cm.$^{-1}$	$\sqrt[4]{\zeta}$	Diff.	Ion	$^2P_{\frac{1}{2}} - {}^2P_{\frac{3}{2}}$ cm.$^{-1}$	$\sqrt[4]{\zeta}$	Diff.
13	*Al*I	112·04	2·939					
				0·781				
14	*Si*II	287·3	3·720					
				·675				
15	*P*III	559·6	4·395					
				·622				
16	*S*IV	950·2	5·017					
				·599				
17	*Cl*V	1 492	5·616		*Cl*I	881	4·923	
				·579				0·636
18	*A*VI	2 210	6·195		*A*II	1 432	5·559	
				·564				·603
19	*K*VII	3 131	6·759		*K*III	2 162	6·162	
				·560				·589
20	*Ca*VIII	4 305	7·319		*Ca*IV	3 115	6·751	
				·552				·576
21	*Sc*IX	5 759	7·871		*Sc*V	4 325	7·327	
								·567
22	*Ti*X	...	...		*Ti*VI	5 825	7·894	
								·558
23	*V*XI	...	...	·543	*V*VII	7 657	8·452	
								·553
24	*Cr*XII	...	...		*Cr*VIII	...	...	
25	*Mn*XIII	...	...		*Mn*IX	...	...	
26	*Fe*XIV	18 852·5 *	10·588		*Fe*X	15 683·2 *	10·112	
		(λ 5303)				(λ 6374)		
				0·538				0·545
27	*Co*XV	...	...		*Co*XI	...	...	
28	*Ni*XVI	27 762 *	11·664		*Ni*XII	23 626 *	11·203	
		(λ 3601)				(λ 4231)		

* Wave-numbers of coronal lines.

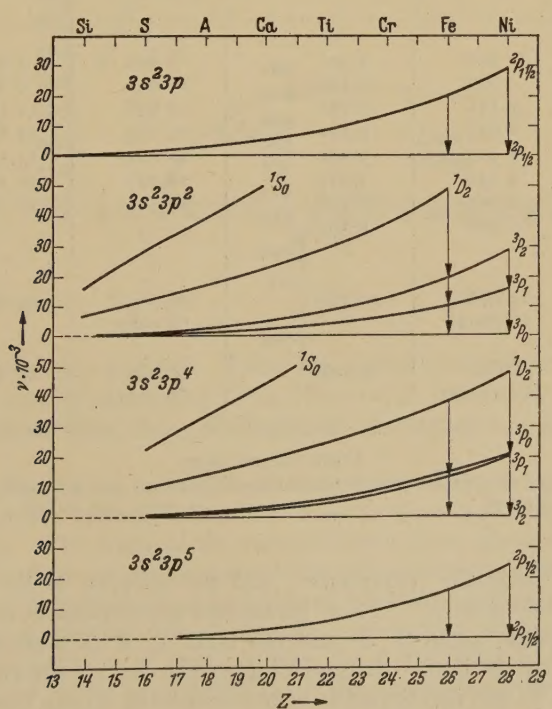

FIG. 1.

TABLE III

Comparison of the Ground-Term Splittings in the Iso-electronic Sequences

$3s^2 3p^2$ (*Si*I, *P*II, . . .)

Ion	$^3P_2 - {}^3P_0$	$\sqrt[4]{\zeta}$		$^1D_2 - {}^3P_2$	F cm.$^{-1}$	
*Si*I	223·3	3·489	712	6 076	5 121	2 000
*P*II	470·3	4·201	642	8 402	7 121	1 821
*S*III	832·5	4·843	609	10 488	8 942	1 727
*Cl*IV	1 341	5·452	590	12 425	10 669	1 669
*A*V	2 028	6·042	573	14 269	12 338	1 635
*K*VI	2 924	6·615	567	16 046	13 973	1 626
*Ca*VII	4 075	7·182	556	17 796	15 599	
*Sc*VIII	5 506	7·738		...	...	
*Ti*IX	...	...		...	...	
*V*X	...	...	547	...	...	1 583
*Cr*XI	...	...		...	...	
*Mn*XII	...	...		...	...	
*Fe*XIII	18 561 *	10·475		29 507 * (λ 3388)	25 098	
*Co*XIV	...	...	541	...	...	
*Ni*XV	27 376 *	11·558		u. violet	...	

$3s^2 3p^4$ (*S*I, *Cl*III, . . .)

Ion	$^3P_1 - {}^3P_2$	$\sqrt[4]{\zeta}$		$^1D_2 - {}^3P_1$	F cm.$^{-1}$	
*S*I	397	4·441	661	8 843	7 523	1 871
*Cl*III	697	5·102	619	10 955	9 394	1 765
*A*III	1 112	5·721	599	12 898	11 159	1 701
*K*IV	1 673	6·320	582	14 713	12 860	1 657
*Ca*V	2 407	6·902	572	16 424	14 517	1 629
*Sc*VI	3 350	7·474	562	18 047	16 146	1 612
*Ti*VII	4 535	8·036	555	19 592	17 758	
*V*VIII	6 006	8·591		...	...	
*Cr*IX	...	...	550	...	...	1 580
*Mn*X	...	...		...	...	
*Fe*XI	12 668 * (λ 7892)	10·240		25 075 * (λ 3987)	24 079	
Co XII	...	...	543	...	...	1 538
*Ni*XIII	19 541 * (λ 5116)	11·326		27 443 * (λ 3643)	27 155	

* From coronal lines.

18 561 = sum of wave-numbers of λ 10747 and λ 10798,

27 376 = „ „ „ λ 6702 and λ 8024.

intervals into the "magnetic" parameter ζ and the intervals of the type $^1D - {}^3P$ into the "electrostatic" parameter F. These parameters are convenient for an extrapolation, the former being proportional to Z^4 and the latter to Z (Z = atomic number). The intervals and the variation in the corresponding parameters are shown in Table III.

These data lead to the identification of the remaining strong coronal lines, λ 3388, λ 10747 and λ 10798. As before, the presence of fainter coronal lines at the accurately

predictable positions for the corresponding nickel transitions support the identifications. In both Tables II and III the numbers inserted for *Fe* and *Ni* are the wave-numbers of coronal lines or in the case of $3s^2 3p^2\ {}^3P_2 - {}^3P_0$ the sum of two such wave-numbers.

The extrapolations leading to the *Fe* and *Ni* identifications are perhaps more clearly illustrated by the diagrams in Figs. 2 and 3, representing in a condensed form the

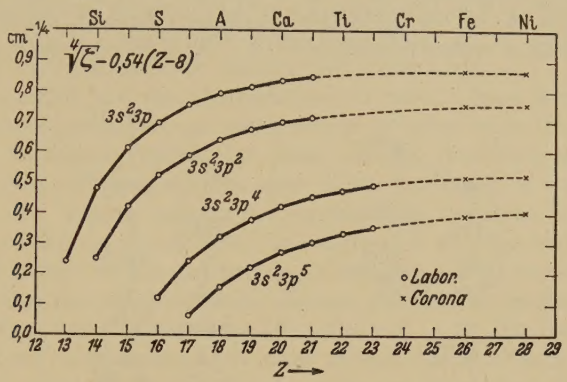

FIG. 2.

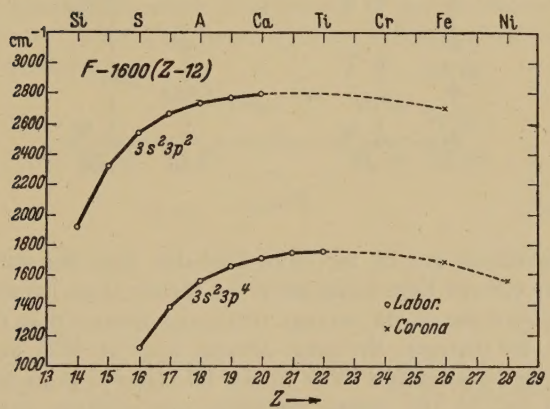

FIG. 3.

variation of the parameters ζ and F. Thus, Fig. 2 gives the evidence for the identification of the five strong iron lines, λλ 5303, 6374, 7892, 10747 and 10798, and of the five corresponding nickel lines, while Fig. 3 illustrates the identification of λ 3388 and two fainter lines.

In this way altogether 13 coronal lines, comprising all the strong lines and more than 95 per cent. of the total intensity of the line emission, have been identified with transitions in *Fe* and *Ni* ions. The origin of the various lines is again illustrated by Fig. 4.

After this exposition of the numerical coincidences of wave-numbers we shall turn to a discussion of the intensities to be expected for various transitions.

The number of quanta emitted through a particular transition is evidently proportional to the number n of ions reaching the initial level of this transition. It is further proportional to the probability A_1 of this transition divided by the sum of the probabilities of all possible ways of de-excitation from the same level. The intensity,

i.e. number of quanta × energy per quantum, is then expressed in arbitrary units by

$$I = vn \frac{A_1}{A_1 + A_2 + A_3 + \ldots + B + C},$$

if A_n stands for the probabilities of spontaneous radiation transitions to lower levels, B for de-excitation by collisions, and C for de-excitation by radiation absorption.

The probabilities A_n of the forbidden transitions here in question can be easily calculated on the basis of recent theoretical work. As a general result it is found (1) that

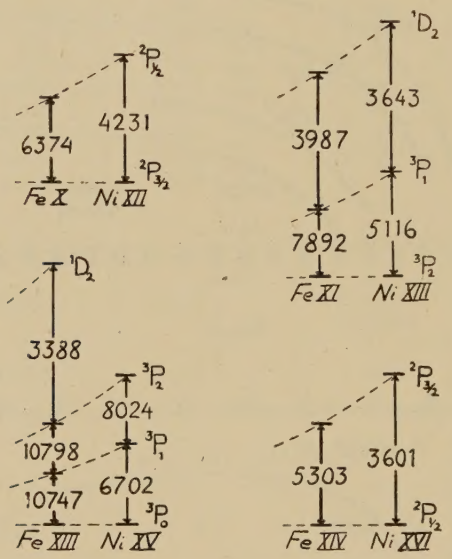

FIG. 4.

magnetic dipole transitions are by far more probable than transitions of the electric quadrupole type, the coronal lines being all of the former type, (2) that the probabilities of the coronal transitions are on the average very much greater than those of the nebular transitions. While, for instance, the main nebular lines of *O*III have transition probabilities about 0·01 per second and those of the *O*II lines λ 3726–29 are much smaller still (about 0·00003 sec.$^{-1}$), the coronal transition probabilities range from 10 to 500 sec.$^{-1}$.

With regard to the spectral structure of the coronal ions and to the intensity distribution in the solar spectrum the process of de-excitation by radiation absorption, represented by C in the intensity formula, should be insignificant. The quantity B, representing the probability of de-excitation through collisions, depends on partly unknown parameters. Theoretical considerations indicate, however, that B should in most cases be smaller than ΣA by a factor of perhaps 1 or 2 powers of ten and thus might be preliminarily neglected in comparison with ΣA. This result seems to be confirmed by the observed intensity ratios of coronal lines belonging to *Fe*XIII. For the discussion of excitation processes in the corona a wave-mechanical calculation of the collisional cross-sections of the ions involved would be very welcome.

The further discussion of line intensities requires a knowledge of the relative number n of ions arriving in a certain metastable level, and we are now forced to decide by what kind of mechanism the ions are transferred into these levels. In this case it is tempting to compare with the similar, thoroughly investigated case of the nebulæ. It was originally assumed by Bowen and has later been confirmed that the metastable levels of the

nebular ions are reached through excitation from the ground level by means of electron collisions. Probably the same process is dominant also in the corona, but it should be remembered that the physical conditions are in some respects quite different and other excitation mechanisms, negligible in the nebulæ, may well be of importance. For instance the intensity of even the strongest coronal lines is an extremely small fraction of the solar radiation of the same wave-length, and consequently an appreciable part of the line emission could well be thought of as a fluorescence excited through the absorption of solar radiation. Such a mechanism is the more likely as the transition probabilities of the "forbidden" coronal lines are comparatively very large.

In order to ascertain that the identifications of coronal lines were at all acceptable with regard to intensities, an attempt was made to calculate what the relative intensities of the identified transitions should be under certain simplified conditions. It was assumed that electron collisions are the cause of the excitation of the metastable levels and de-excitation by means other than spontaneous radiation is negligible. The number of ions arriving in a given level was derived in analogy with the treatment by Menzel, Aller and Hebb * of the similar case of forbidden *O*III in nebulæ. On further assumptions regarding the collisional cross-sections, the electron temperature ($=250\,000°$) and the relative abundance of elements ($Fe : Ni = 100 : 5{\cdot}2$), the intensity in arbitrary units of any one of the coronal transitions in the various *Fe* and *Ni* ions could be expressed as a product of a calculable factor and the unknown percentage R of the particular ionization stage.† The figure for the electron temperature is rather a guess, but it should be remarked that the result is very little affected as long as the temperature is assumed to be very high, an assumption which is supported by several independent facts. The calculation could be much improved, however, if the theoretical values for the collisional cross-sections were available.

TABLE IV

"Calculated" and Observed Intensities ‡

Configuration	Transition	Iron				Nickel			
		Ion	I'	I (obs.)	I (obs.)/I'	Ion	I'	I (obs.)	I (obs.)/I'
$3s^2 3p^5$	$^2P_{\frac{1}{2}} - {}^2P_{\frac{3}{2}}$	*Fe*X	43	8·1 (18)	0·19 (0·42)	*Ni*XII	3·1	2·6 ...	0·84 ...
$3s^2 3p^4$	$^1D_2 - {}^3P_1$	*Fe*XI	14	0·7 ...	0·05 ...	*Ni*XIII	0·5	faint ...	... ...
	$^3P_1 - {}^3P_2$	,,	70	... (13)	... (0·19)		5·2	4·3 (2·2)	0·83 (0·42)
$3s^2 3p^2$	$^1D_2 - {}^3P_2$	*Fe*XIII	87	16 ...	0·18 ...	*Ni*XV	4·5	u. violet ...	... ...
	$^3P_2 - {}^3P_1$	,,	89	... (35)	... (0·39)		5·6	... (0·5)	... (0·09)
	$^3P_1 - {}^3P_0$	,,	160	... (55)	... (0·34)		12	5·4 (2·0)	0·45 (0·17)
$3s^2 3p$	$^2P_{\frac{3}{2}} - {}^2P_{\frac{1}{2}}$	*Fe*XIV	100	100 (100)	1 (1)	*Ni*XVI	7·2	2·1 ...	0·29 ...

In Table IV the factors I' thus calculated are collected and these "calculated intensities" are compared with the observed intensities of the coronal lines. With regard to the considerable uncertainty in both calculations and observations, the agreement is satisfactory. The table contains only the observed transitions, but it should be pointed out that the absence of the other possible transitions within the same configurations is in accord with negligible values for the calculated intensities.

* *Ap. J.*, **93**, 230, 1941.

† See *Z. Astrophys.*, **22**, 30, 1942.

‡ The observed intensities are taken from Grotrian or based on Lyot's observations (figures in parentheses).

The ratio I(obs.)/I' should be proportional to the abundance R of the different ionization stages and thus provide a quantitative indication as to the degree of ionization of the coronal gas. The result is illustrated by Fig. 5 where this ratio is plotted for selected pairs of *Fe* and *Ni* lines. The relative abundance of ions evidently increases from *Fe*X to *Ni*XII, reaches a maximum around the XIII ionization stage and decreases again from *Fe*XIV to *Ni*XVI. It should be remarked here that lines of *Fe*XII cannot be expected because of unfavourable positions of the metastable levels in that ion.

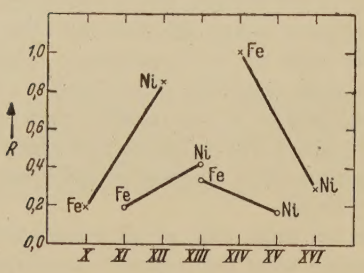

FIG. 5.

This discussion shows in any case that the identifications of the iron and nickel lines are compatible with calculable transition probabilities and known cosmic abundances of these elements.

A few words may be said about some additional identifications. In the first place we have to consider ions of the configurations $2s^2 2p^n$ analogous to those of $3s^2 3p^n$ already treated. In the sequence *F*I, *Ne*II, etc., with the electron configuration $2s^2 2p^5$ the splitting of the ground term ^{2}P is directly determined from measurements in the extreme ultra-violet spectrum of the vacuum spark up to *Ca*XII as shown in Table V.

TABLE V

Ground-Term Splittings in the Iso-electronic Sequence $2s^2 2p^5$ (*F*I, *Ne*II, . . .); $\zeta = \frac{2}{3}\Delta\nu$

Z	Ion	$^2P_{\frac{1}{2}} - {}^2P_{\frac{3}{2}}$ cm.$^{-1}$	$\sqrt[4]{\zeta}$	Diff.
9	*F*I	404·0	4·051	
10	*Ne*II	782	4·778	0·727
11	*Na*III	1 364	5·491	·713
12	*Mg*IV	2 226	6·207	·716
13	*Al*V	3 440	6·920	·713
14	*Si*VI	5 097	7·635	·715
15	*P*VII	7 268	8·343	·708
16	*S*VIII	10 081	9·054	·711
17	*Cl*IX	13 641	9·765	·711
18	*A*X	(18 063)	...	·710
19	*K*XI	23 475	11·185	0·710
20	*Ca*XII	30 028	11·895	

As previously mentioned, the *Ca*XII transition has been identified with the coronal line λ 3328. Furthermore, the separation interpolated for *A*X fits exactly with the wave-number of coronal line λ 5536. There is, however, no trace of a coronal line corresponding to the separation of *K*XI. As the conditions for an appearance of this line should be more favourable than for other possible transitions in potassium, this element is evidently not present in the coronal spectrum. The absence of potassium was rather puzzling with regard to the abundance ratio $K : Ca = 1{\cdot}2$ from Russell's well-known analysis of the solar atmosphere. On the other hand a ratio $K : Ca = 0{\cdot}12$ corresponding to the average composition of meteorites (Goldschmidt) would explain the absence of potassium and consequently I was led in my first note on the coronal lines to suggest a meteoric origin for the coronal matter. Recently, however, Bengt Strömgren found with a refined method exactly the meteoric ratio 0·12 for the solar

atmosphere.* The argument for a meteoric origin of the coronal matter is therefore no longer valid. Indeed, several facts rather suggest that the figures from the analyses of meteorites could be generally accepted as the most accurate representation of the cosmic distribution of metals.

TABLE VI

Relative Atomic Abundance of Metals in Meteorites

Z	19	20	22	24	25	26	27	28
Atom	*K*	*Ca* *	*Ti*	*Cr*	*Mn*	*Fe* *	*Co*	*Ni* *
Abundance	0·8	6·4	0·5	1·3	0·7	100	0·4	5·2

* Observed in the corona.

The collection of these abundance data in Table VI, comprising all elements heavier than potassium that are present in meteorites by more than 1 atom per 1000 iron atoms, gives an obvious reason for the selection of elements observed in the corona.

TABLE VII

The Emission Lines in the Solar Corona: Observational Data and Identifications

λ	ν cm.$^{-1}$	Intensity		Identification		A_m sec.$^{-1}$	E.P.	I.P.
3 328	30 039	1·0	...	*Ca*XII	$2s^2 2p^5$, $^2P_{\frac{1}{2}} - {}^2P_{\frac{3}{2}}$	488	3·72	589
3 388·1	29 507	16	...	*Fe*XIII	$3s^2 3p^2$, $^1D_2 - {}^3P_2$	87	5·96	325
3 454·1	28 943	2·3	...		...	...	...	...
3 601·0	27 762	2·1	...	*Ni*XVI	$3s^2 3p$, $^2P_{\frac{3}{2}} - {}^2P_{\frac{1}{2}}$	193	3·44	455
3 642·9	27 443	...	...	*Ni*XIII	$3s^2 3p^4$, $^1D_2 - {}^3P_1$	18	5·82	350
3 800·8	26 303	...	...		...	...	...	...
3 986·9	25 075	0·7	...	*Fe*XI	$3s^2 3p^4$, $^1D_2 - {}^3P_1$	9·5	4·68	261
4 086·3	24 465	1·0	...	*Ca*XIII	$2s^2 2p^4$, $^3P_1 - {}^3P_2$	319	3·03	655
4 231·4	23 626	2·6	...	*Ni*XII	$3s^2 3p^5$, $^2P_{\frac{1}{2}} - {}^2P_{\frac{3}{2}}$	237	2·93	318
4 311	23 190	...	...		...	...	...	...
4 359	22 935	...	...	?*A*XIV	$2s^2 2p$, $^2P_{\frac{3}{2}} - {}^2P_{\frac{1}{2}}$	108	2·84	682
4 567	21 890	1·1	...		...	...	...	...
5 116·03	19 541·0	4·3	2·2	*Ni*XIII	$3s^2 3p^4$, $^3P_1 - {}^3P_2$	157	2·42	350
5 302·86	18 852·5	100	100	*Fe*XIV	$3s^2 3p$, $^2P_{\frac{3}{2}} - {}^2P_{\frac{1}{2}}$	60	2·34	355
5 536	18 059	...	...	*A*X	$2s^2 2p^5$, $^2P_{\frac{1}{2}} - {}^2P_{\frac{3}{2}}$	106	2·24	421
5 694·42	17 556·2	...	1·2	?*Ca*XV	$2s^2 2p^2$, $^3P_1 - {}^3P_0$	95	2·18	814
6 374·51	15 683·2	8·1	18	*Fe*X	$3s^2 3p^5$, $^2P_{\frac{1}{2}} - {}^2P_{\frac{3}{2}}$	69	1·94	233
6 701·83	14 917·2	5·4	2·0	*Ni*XV	$3s^2 3p^2$, $^3P_1 - {}^3P_0$	57	1·85	422
7 059·62	14 161·2	...	2·2	*Fe*XV	$3s3p$, $^3P_2 - {}^3P_1$	...	31·7	390
7 891·94	12 667·7	...	13	*Fe*XI	$3s^2 3p^4$, $^3P_1 - {}^3P_2$	44	1·57	261
8 024·21	12 458·9	...	0·5	*Ni*XV	$3s^2 3p^2$, $^3P_2 - {}^3P_1$	22	3·39	422
10 746·80	9 302·5	...	55	*Fe*XIII	$3s^2 3p^2$, $^3P_1 - {}^3P_0$	14	1·15	325
10 797·95	9 258·5	...	35	*Fe*XIII	$3s^2 3p^2$, $^3P_2 - {}^3P_1$	9·7	2·30	325

Table VII gives the complete set of coronal lines with observational data and identifications. The columns contain: wave-length λ, wave-number ν, intensity according to Grotrian and Lyot, identified transition in the usual spectroscopic notation, transition probability A_m, excitation energy E.P. in electron volts, and ionization energy I.P. for the next preceding ion.

The line λ 5694, tentatively identified as *Ca*XV, deserves a special remark. This

* *Festskrift för Elis Strömgren*, Copenhagen, 1940.

line was first observed by Lyot and is normally very faint, but on one occasion it was observed by Waldmeier over a large active spot group with an intensity three times greater than that of the green coronal line. This exceptional behaviour of λ 5694 could well be compatible with the proposed identification, which implies an ionization potential (>800 volts) decidedly higher than for any of the other ions identified. To prove the identification, however, a second transition in the same ion must appear with the same behaviour and a similar intensity within the visible spectrum. Such a line has not yet been observed.

According to the explanation now described the essential part of the coronal lines is due to iron, nickel, calcium and argon atoms deprived of 10 to 15 electrons, *i.e.* about half of their normal electron envelope. The discovery of this enormously high stage of ionization has obviously introduced a new argument in the discussion of solar phenomena. Several attempts to give the established facts a physical explanation have already been made.

Let us first recall some of the more obvious arguments for the existence of a very high temperature in the corona:

1. The high mean stage of ionization as revealed by the emission lines.
2. The breadth of the emission lines, if due to thermal Doppler effect. The broadening of the lines might also be caused by macroscopic irregular motions (turbulence) or radial motions of the matter.
3. The blurring out of the Fraunhofer lines in the continuous spectrum of the inner corona, assumed to be an effect of the velocities of the scattering electrons.
4. The absence of the Balmer lines in the emission line spectrum of the corona, explained by the electrons being too fast to be captured by the protons.
5. Dynamical considerations showing that great thermal velocities are necessary to balance the gravitational forces in order to explain the observed density gradient of the corona.

All these observations point to temperatures higher than a quarter of a million degrees.

Independently of the identification of the coronal lines, Alfvén * came to the conclusion that the corona might consist altogether of particles with very high energy and derived from the density function a temperature of about one million degrees. On certain assumptions Alfvén finds that the energy necessary to maintain this high temperature of the corona would be about 10^{-5} of the total energy radiated by the Sun, and that the total energy contained in the corona would be produced in about two hours. The "heating" mechanism conceived by Alfvén is described in the following way:

"Motion of solar matter in magnetic fields on the Sun, especially the vortical motion in a sunspot must bring about potential differences between different points of the solar surface, and it was shown [in a previous paper] that under certain conditions this gives rise to discharges above the surface of the Sun. Calculations indicate that the electromotive force can be as high as 10^7 volts, so that even if charged particles are usually accelerated only by a small fraction of this potential, they attain rather high energies.

"The process is most conspicuous in the prominences, where consequently we can expect a very intense production of high energy particles. As the mechanism is of a very general character the same process is likely to take place very frequently on a smaller scale. If—as many authors mean—we can regard the chromosphere as a multitude of small prominences, it is likely that a production of high energy particles takes place almost everywhere on the solar surface or in some layer above it."

A different view on the origin of the highly ionized particles in the corona was taken by Menzel † by the suggestion that the coronal matter was ejected from the hot interior of the Sun through holes and cracks in the solar surface. A somewhat similar opinion

* *Arkiv Mat. Astr. Fys.*, **27** A, No. 25, 1941.
† *The Telescope*, **8**, 65, 1941.

is expressed by Vegard * in a recent paper. According to his theory "the highly ionized heavy ions present in the corona come from the Sun's deeper layers and are driven away from the Sun at great speed through the electric fields resulting from photo-electric effect produced by soft X-rays".

Finally, a quite different explanation has been put forward by Saha †, who suggests that the highly ionized atoms emitting the coronal lines are the fragments of a kind of nuclear fission, similar to the uranium fission, occurring somewhere near the solar surface.

Before the various suggestions have been more thoroughly examined it would be unwise to judge in favour of the one or the other. In that respect the physical explanation of the solar corona still remains a problem.

Appendix

As mentioned above the starting-point for the identification of the coronal lines was given by Grotrian. His letter of 1937 February 13 reads (in translation) as follows:—

Dear Dr. Edlén,

According to your latest article on the spectra of highly ionized atoms, the separation of the ground levels of the *Fe*X-spectrum is $15{\cdot}69 \times 10^3$ cm.$^{-1}$. The wave-number of the well-known red coronal line λ 6374·75 is 15 682·3 cm.$^{-1}$. Thus, the wave-numbers agree within the limits of error of your present measurements. This may naturally be a chance coincidence. However, I would like to ask: would it be possible for you to increase the accuracy of your measurements so far as to make possible the decision whether the coincidence has a real significance or not?

Also the separation of the ground levels $^3P_2 - {}^3P_1$ of the *Fe*XI-spectrum, for which you give the value $12{\cdot}68 \times 10^3$ cm.$^{-1}$, agrees within the limits of error with the wave-number of the infra-red coronal line λ 7891·6; $\nu = 12\ 668{\cdot}2$, recently observed by Lyot. Other forbidden transitions of this spectrum, which may be expected, are obviously not manifested as coronal lines. Thus I am very sceptical whether the above-mentioned numerical agreements mean more than chance coincidences. Perhaps it will be worth while, however, to investigate the question. I should be glad to hear your opinion on this matter and remain with my best greetings,

Yours sincerely,

W. Grotrian.

* *Geofys. Publ.*, **16**, No. 1, 1944.

† *Proc. Phys. Soc.*, **57**, 271, 1945; see also P. Swings, *P.A.S.P.*, **57**, 117, 1945.

AN IMPROVED TYPE OF SCHMIDT CAMERA

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Summary

By substituting for the aspheric plate of the Schmidt camera a combination of an aspheric plate and a meniscus lens concentric with the mirror surface, a system is obtained whose performance at a nominal aperture of $f/1{\cdot}2$ is much superior both to that of the Schmidt camera and to that of the Maksutov system with spherical surfaces. The colour-error introduced by the meniscus is compensated by making the plate in the form of a cementable doublet with a spherical interface. The residual colour-error is comparable with that of the classical Schmidt camera.

An analysis is made of the off-axis errors of the system and it is shown that, working over an 18° diameter field, the system sends 97 per cent. of the light of each image into a confusion-circle of angular diameter 12 seconds of arc. Over a 30° field, 97 per cent. of the light of each image is sent into a confusion circle 21 seconds of arc in diameter. The remaining 3 per cent. of the light does not sensibly affect the resolution, but it can be eliminated if desired, by introducing a pair of additional stops close to the aspheric surface, at the cost of a 6 per cent. light-loss by vignetting at the edge of the field. The overall error-spreads are thereby reduced to one tenth of those of a classical Schmidt camera in the first case and one sixteenth in the second.

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1. *Introduction.*—The first major advance towards the practical application of aspheric surfaces to photographic objectives was made fifteen years ago by Bernhard Schmidt when he devised and constructed the camera which bears his name. It is true that paraboloidal mirrors for astronomical telescopes had been in general use for more than one hundred and fifty years, but such mirrors can only be used over very small fields, owing to their heavy off-axis coma. It is precisely the idea of the Schmidt camera to remove this coma by carrying out the aspherical "figuring", not on the surface of the mirror, but on a plane-parallel glass plate placed in front of it and located at its centre of curvature. The result is an anastigmatic system, consisting of a spherical mirror and an aspheric plate (see Fig. 1), whose performance at high apertures is so much superior to that of orthodox camera lenses that it is rapidly coming into general use in astronomical work, in spite of the disadvantages of its curved field and the difficulty of constructing the aspheric plate to the required accuracy.

The remarkable success of the Schmidt camera in practice and its rapidly increasing popularity among astronomers, especially in the United States, soon attracted the notice of orthodox lens-designers, and during the following eight years several lens-mirror systems were designed in which converging power was supplied by a concave spherical reflecting surface, while a lens-combination, situated some distance in front of the mirror and employing only spherical surfaces, took over the functions of the original Schmidt plate. But it was not until 1944 that it was pointed out by D. D. Maksutov that a single thin meniscus lens, its surfaces nearly concentric with the surface of the mirror M (see

Fig. 2), can replace the aspheric plate in Schmidt cameras of moderate speed and aperture.* Of course the construction and mounting of a thin, sharply curved meniscus lens of large size brings its own problems, but it may well be found easier, at speeds of $f/2{\cdot}5$ or $f/3$ and apertures up to two or three feet, to obtain satisfactory photographic definition over a 12° field with a Maksutov system than with a Schmidt.

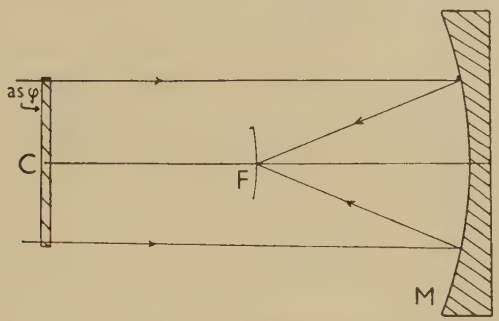


FIG. 1.—*Schmidt system.*

The monocentric Maksutov system, in which the surfaces of the meniscus lens are strictly concentric with that of the mirror, has the advantage that the aberrations do not increase as we go off axis, but it suffers from a considerable amount of primary colour-error, as well as from zonal aberration over the whole field.

In the present paper we consider what can be achieved by combining the two ideas of Maksutov and of Schmidt, and show that it is possible to design a system whose performance at a speed of $f/1{\cdot}2$ is much superior to that of both the Schmidt and of the meniscus system with spherical surfaces.† We shall refer to this system (shown in

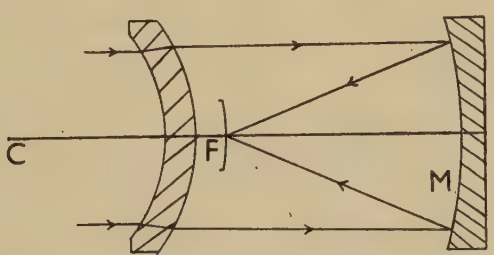


FIG. 2.—*Maksutov system.*

Fig. 3) as a meniscus Schmidt; its mode of operation can best be described by comparing it with the classical Schmidt. In the classical Schmidt camera, converging power is provided by the spherical concave mirror at the cost of introducing spherical aberration; no other aberrations affecting image quality are introduced by the mirror provided it is used with a stop at the centre of curvature C. The function of the Schmidt plate is to correct the spherical aberration; it does so at the cost of introducing certain higher order off-axis aberrations and a small amount of colour-error. These constitute the residual errors of the system. If we choose to analyse the spherical aberration of the mirror into

* *J. Opt. Soc. Amer.*, 34, 270–284, 1944. Mention should also be made of a system by D. Gabor (British Patent Specification, 544, 694; 1940) based on the same idea and employing an annular aperture.

† Maksutov has considered meniscus systems with retouched surfaces in the paper already referred to; so far as we are aware, no discussion of the off-axis properties of such systems has been published. By figuring the mirror and one of the meniscus surfaces, systems are obtained which stand in much the same relation to the meniscus Schmidt as do the Ritchey-Chrétien two-mirror aplanats to the classical Schmidt system.

third order (*i.e.* primary) aberration, fifth order aberration and so on, and to analyse the equation $s=f(r)$ of the plate profile (see Fig. 4A) into a corresponding series-equation

$$s=br^4+cr^6+\ldots,$$

then the effect of the plate can be described by saying that the term br^4 annuls the primary spherical aberration without introducing any third order errors *, but at the cost of

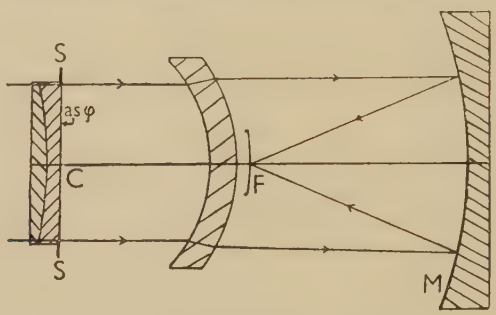


FIG. 3.—*Meniscus Schmidt system.*

introducing two fifth order errors, namely lateral spherical aberration and higher astigmatism, into the off-axis images. Further higher order errors are introduced, in small amount, by the terms $cr^6+\ldots$ In the classical Schmidt, therefore, the fifth and higher order off-axis errors are not corrected, and the good performance of the system is due to their intrinsic smallness.

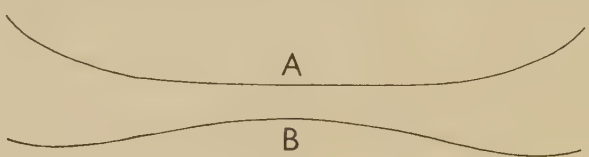


FIG. 4.

A.—*Paraxially flat Schmidt plate,* $s=br^4+cr^6+\ldots$

B.—*Colour-minimised Schmidt plate,* $s=ar^2+br^4+cr^6+\ldots$

In the meniscus Schmidt, we can first correct the Seidel aberrations of the mirror by a suitably chosen concentric meniscus. Used with a stop at C (Fig. 3), the system consisting of these two components suffers from higher order spherical aberration, uniform all over the field, but is free from all other errors affecting image-quality, except colour-error, which we disregard for the present. We can now remove the residual errors in the on-axis image by a suitably figured aspheric plate located in the aperture-stop. The profile of this plate is represented by an equation of the form

$$s=cr^6+\ldots,$$

since the primary spherical aberration of the mirror has already been corrected by the meniscus.

By the same argument as is used to establish the Seidel anastigmatism of the classical Schmidt with paraxially flat plate, it follows in the present case that the images formed on the spherical field surface, of centre C and radius CF (Fig. 3), are free from both third- and fifth-order errors. The residual errors of the system are all of the seventh and higher orders, and so can be expected to be very small. The small amount of primary

* If the plate were placed anywhere along the axis except at C, the term br^4 would introduce primary coma and also primary astigmatism into the off-axis images, as may be seen at once by considering the "plate diagram" of the system.

chromatism introduced by the meniscus can be corrected, without introducing too great errors elsewhere, by making the plate in the form of a cementable doublet, with a spherical interface, from two glasses of the same mean refractive index but of different dispersions.

In the actual designing of the Schmidt and of the meniscus Schmidt, the above idea is applied in a slightly modified form. In the classical Schmidt, the plate profile which gives the best performance is not that which exactly compensates the third-order errors of the mirror; better results are obtained with a profile, convex in the centre and concave near the edge, which gives on-axis stigmatism and also minimises the profile-slope over the whole plate (see Fig. 4B). A profile satisfying this condition is usually recommended because it minimises the colour-error of the system, but an equally valid reason for preferring it to the paraxially flat type is that it introduces less lateral spherical aberration and less off-axis astigmatism.

Similar considerations apply in the case of the meniscus Schmidt. Instead of choosing a meniscus which compensates exactly the primary spherical aberration of the mirror, we do better to choose one which reduces to a minimum the angular errors in the outer parts of the aperture. The profile of the correcting plate, which is designed to give axial stigmatism, is in this way made more favourable from the point of view of off-axis aberrations, and the colour properties of the plate are also improved.

2. *Design of an f/1·2 Meniscus Schmidt.*—The design calculations fall into three stages.

In the first stage a strictly concentric meniscus of suitable thickness was found by trial so that the angular deviations in *d*-light ($\lambda = 5875$) are as small as possible over the whole aperture. The radius of curvature of the spherical mirror was taken as 30 inches, and this implies an aperture diameter of approximately 12 inches. The thickness of the meniscus was fixed at 2 inches, leaving it with only one degree of freedom. Not much is lost by this arbitrary restriction, since it appears probable that the thickness of a 12-inch meniscus should not be very different from 2 inches for practical reasons.* We suppose the glass of the meniscus to be a hard crown with the refractive indices

$$n_C = 1{\cdot}51392, \qquad n_d = 1{\cdot}51654, \qquad n_F = 1{\cdot}52247.$$

This type of glass is less difficult than many to produce in large homogeneous plates. It seems not unlikely, however, that a selected piece of white plate glass, drop-moulded and carefully reannealed, would give equally good results at a much lower cost.

The values

$$r_1 = 10{\cdot}810 \text{ inches}, \qquad r_2 = 12{\cdot}810 \text{ inches}$$

for the radii of the meniscus surfaces were found to bring a parallel beam of *d*-light to a focus at F (Fig. 3), where CF = 13·9995 inches, with small and satisfactorily balanced angular errors η. In the actual computation, the rays are traced out from F at selected angles θ with the axis, and η measures their inclination to the axis after passing through the meniscus. Table I gives the values of η, in seconds of arc, for selected values of $\sin\theta$ and selected wave-lengths. They are shown graphically, in Fig. 5, for the *d*-line ($\lambda = 5875$).

Fig. 6 shows the colour-error introduced by the meniscus; it is obtained by graphing $\eta_d - \eta_F$ and $\eta_d - \eta_C$ against $\sin\theta$. The two curves are very nearly straight lines through the origin; this shows that the colour effect of the meniscus consists almost entirely in a small chromatic variation of focal length (about 0·012 inch between C and F) which can be compensated by the introduction of a spherical interface into the figured plate, or by slightly convexing its front surface. In the present paper we have adopted the first expedient, but the greater simplicity of the second commends it in cases where a lower standard of performance can be tolerated.

* If the thickness is much below two inches, flexure of the glass may be expected to become troublesome; if much above, flexure of the mounting and absorption of light by the glass.

The second stage of the calculation consists in determining the plate-profile so as to correct the residual errors of the axial image in *d*-light. We suppose that the aspheric

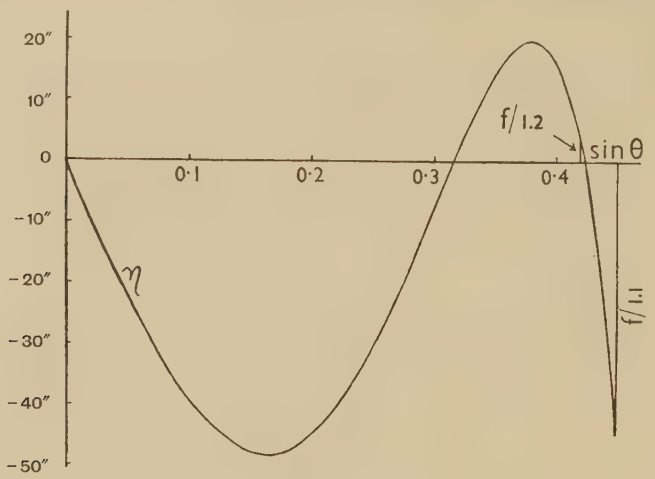


FIG. 5.

TABLE I

sin θ	η (in seconds of arc) C-line	*d*-line	F-line
0·02		− 9·05	
4		17·57	
6		25·67	
8		32·82	
0·10		−38·87	− 49·75
2		43·48	
4		46·57	
6		47·91	
8		47·45	
0·20	−35·09	−44·97	− 67·22
2		40·76	
4		34·64	− 61·80
6		26·82	
8		17·97	− 50·28
0·30	+ 7·36	− 8·19	− 42·22
2		+ 1·56	
4		10·33	
6		16·91	− 26·82
8		19·53	− 27·33
0·40	+38·33	+16·05	− 34·08
2		+ 3·64	
4		−21·25	
6	−36·25	−63·34	−124·07

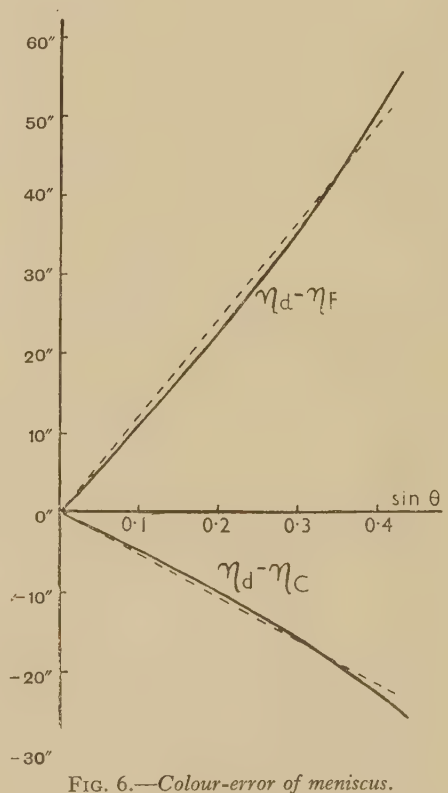


FIG. 6.—*Colour-error of meniscus.*

surface of the plate is turned towards the mirror, and that n_d' is the refractive index for *d*-light of the glass on this side. Then if s is the amount by which the plate-thickness at the zone of radius r falls below its thickness at the centre, the equations

$$(n_d' - 1)\frac{ds}{dr} = \frac{\pi}{648{,}000}\eta_d(r); \qquad r = 13{\cdot}9995 \sin\theta$$

enable s to be calculated as a function of r from Table I by numerical integration. The result is shown graphically in Fig. 7, from which we see that the departure of the aspheric surface from the flat is nowhere greater than 28 fringes, i.e. less than one-twelfth of that of an ordinary twelve-inch $f/1{\cdot}2$ Schmidt plate. Because of its "softer" turn-up edge, and also because of its much smaller overall asphericity, the aspheric plate of a meniscus Schmidt should be a good deal easier to figure than an ordinary Schmidt plate.

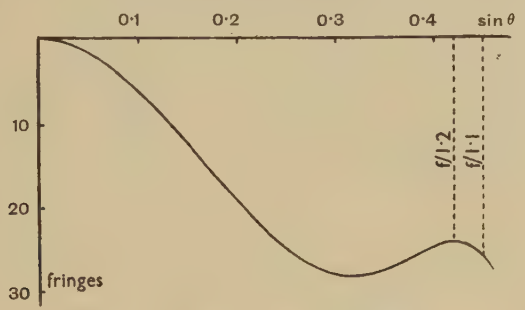


FIG. 7.—*Profile of corrector plate.*

The last stage of the design-calculation consists in choosing the glasses of the plate and determining the interface radius so as to correct the chromatic variation of focus introduced by the meniscus. From Fig. 6 we see that by choosing two glasses with reasonably well-matched dispersion runs (for example, a hard crown and a telescope flint) the residual colour-spreads from the meniscus can be reduced to about 5 seconds of arc over the whole spectrum. In particular, the glasses HC 525588 and TF 525512 in Chance's catalogue allow the chromatic variation of focus to be corrected in this way. The radius of curvature of the interface comes out at approximately 48 inches *, and its off-axis colour-aberrations need to be taken into account in assessing the performance of the system. By an application of Seidel theory (which is adequate for the purpose) we find that these off-axis colour aberrations act, in the main, in opposition to the residual sphero-chromatic aberration of the system, and that at each point of an 18° diameter field the colour-confusion patch corresponding to the spectral range C–F can be covered by a circle 8 seconds of arc in angular diameter.† This limit corresponds to about 0·0005 inch in the field surface. Near the axis, the residual colour-spread is approximately the same as that in a classical Schmidt camera of the same speed and aperture. Over a 30° diameter field the corresponding upper limit to the colour-spreads is 11 seconds of arc.

3. *Monochromatic Aberrations.*—We next apply the methods of C. Carathéodory ‡ to discuss the off-axis aberrations of the system in *d*-light. A knowledge of these and of the colour errors makes it possible to estimate the performance of the system and the size of its useful field.

From Fig. 3 we see that, in order to determine the image-errors at an angular distance ϕ from the centre of the field, it it sufficient to calculate the effect on the axial image of tilting the plate, together with the aperture-stop S, through an angle ϕ about an axis through C in the plane of the plate. Taking Cartesian co-ordinates x, y in the plane of the aperture-stop, with origin at its centre C, we suppose that the plate is tilted through

* The exact position of the interface does not need to be specified, as its effect is substantially the same in all positions.

† Including 1·7 seconds of arc arising from the asphericity of the plate.

‡ "Elementare Theorie des Spiegelsystems von Bernhardt Schmidt", *Hamb. Math. Einzelschr.*, **28**, 36 pp., 1940. We have unfortunately not been able to gain access to a copy of this paper.

an angle ϕ about the axis Oy. The plate-thickness $T(x, y)$ is very nearly constant and the slopes $\partial T/\partial x$ and $\partial T/\partial y$ of its aspheric surface in the x- and y-directions are very small, so that their squares can be neglected.

Now consider a ray, initially parallel to the axis of the system and passing through the point (x, y) of the aspheric plate-surface (which agrees to within 28 fringes with the plane of the aperture-stop). The angular deviations which this ray receives from the untilted plate amount to

$$(n-1)\partial T/\partial x, \qquad (n-1)\partial T/\partial y \tag{3.1}$$

radians in the x- and y-directions respectively. These deviations produce an error-free axial image, and it is the changes which they undergo when the plate is tilted that give rise to the monochromatic aberrations. Two main factors operate to produce these changes. In the first place, the tilted plate is traversed obliquely by the rays, and this increases the angular deviations associated with each point (x, y) of the plate surface. The new values are given, when $\sin^4\phi$ is small enough to be neglected, by the expressions*

$$(n-1)\frac{\partial T}{\partial x}\left[1+\frac{n+1}{2n}\sin^2\phi\right], \qquad (n-1)\frac{\partial T}{\partial y}\left[1+\frac{1}{2n}\sin^2\phi\right]. \tag{3.2}$$

In the second place, owing to the tilt of the plate, a point on the front surface of the meniscus which formerly received light from the point (x, y) on the aspheric surface of the plate now receives it from the point $(x\sec\phi, y)$. This causes a further change, of amount

$$(n-1)x\frac{\partial^2 T}{\partial x^2}(\sec\phi-1), \qquad (n-1)x\frac{\partial^2 T}{\partial x\partial y}(\sec\phi-1),$$

in the angular deviations. Other effects of the plate-tilt are negligibly small and we therefore obtain the expressions

$$(n-1)\sin^2\phi\left[\frac{n+1}{2n}\frac{\partial T}{\partial x}+\frac{x}{2}\frac{\partial^2 T}{\partial x^2}\right], \qquad (n-1)\sin^2\phi\left[\frac{1}{2n}\frac{\partial T}{\partial y}+\frac{x}{2}\frac{\partial^2 T}{\partial x\partial y}\right] \tag{3.3}$$

for the components of angular aberration in the ray through the point (x, y) of the entry pupil. The linear deviation-components (ξ, η) in the field-surface are obtained on multiplying these expressions by the focal length f, and so are given by the equation

$$\xi+i\eta=(n-1)f\sin^2\phi\left(\frac{n+1}{2n}\frac{\partial T}{\partial x}+\frac{x}{2}\frac{\partial^2 T}{\partial x^2}+\frac{i}{2n}\frac{\partial T}{\partial y}+\frac{ix}{2}\frac{\partial^2 T}{\partial x\partial y}\right). \tag{3.4}$$

Since the plate is a solid of revolution, we write $T=T(x^2+y^2)=T(r^2)$ in (3.4), which then takes the form

$$\xi+i\eta=(n-1)f\sin^2\phi\left\{2xT'(r^2)+(x+iy)\left(\frac{1}{n}T'(r^2)+2x^2T''(r^2)\right)\right\}. \tag{3.5}$$

Change of Notation.—It is convenient to introduce, in place of $T(r^2)$, the function

$$P(t^2)=\frac{648000(n-1)}{\pi f}[T(r^2)-T(0)],$$

whose t-derivative

$$\eta(t)=2tP'(t^2)$$

measures, in seconds of arc, the radial deviation imposed by the plate on a ray meeting

* If $\sin^4\phi$ is not negligible, the deviations are more accurately given by the expressions

$$\frac{\partial T}{\partial x}[(n^2-\sin^2\phi)^{\frac{1}{2}}-\cos\phi], \qquad \frac{\partial T}{\partial y}[\sec\phi(n^2-\sin^2\phi)^{\frac{1}{2}}-1];$$

the approximations (3·2) are sufficiently accurate for the purpose of the present paper, however.

it normally at the zone of radius $r=ft$. Such a ray comes in to the focus at an angle θ given by the equation $t=\sin\theta$. We also write

$$\delta X=\frac{648000}{\pi f}\xi, \qquad \delta Y=\frac{648000}{\pi f}\eta,$$

so that δX, δY are the apparent angular aberrations, measured in seconds of arc, in the incoming ray. (3.5) is then equivalent to the equation

$$\delta X+i\delta Y=\sin^2\phi\left\{2\alpha uP'(t^2)+(u+iv)\left(\frac{\alpha}{n}P'(t^2)+2\alpha^3u^2P''(t^2)\right)\right\}$$

or to

$$\delta X+i\delta Y=\sin^2\phi\left\{\frac{\alpha}{i}u\eta(t)+(u+iv)\left(\frac{\alpha}{2nt}\eta(t)+\frac{\alpha^3u^2}{2t^3}(t\eta'(t)-\eta(t))\right)\right\}, \tag{3.6}$$

where $\alpha=R/f=0{\cdot}42$ is the numerical aperture of the system, R the radius of the aperture-stop, $u=x/R$, $v=y/R$, $t^2=\alpha^2(u^2+v^2)$. The edge of the aperture-stop is now the circle $u^2+v^2=1$.

Polynomial Approximations.—The computations are somewhat simplified if we approximate to the "optical plate-slope" $\eta(t)$ by a polynomial

$$\eta^*(t)=At+Bt^3+Ct^5+Dt^7 \tag{3.7}$$

in such a way as to minimise the error

$$\epsilon=\max_{0\leqslant t\leqslant\alpha}\ |\ \eta(t)-\eta^*(t)\ |.$$

Then the equation

$$P^*(t^2)=\int_0^t\eta^*(u)du \tag{3.8}$$

defines a polynomial approximation $P^*(t^2)$ to the function $P(t)$ and so to the plate profile.

In the present case, the polynomial

$$\eta^*(t)=-457t+6350t^3-13340t^5-43430t^7 \tag{3.9}$$

agrees with the function η of Table I to within $0{\cdot}''8$ of arc throughout the range $0<t<0{\cdot}42$; the residual differences are shown in Fig. 8. From (3.6) we can then calculate the quantity

$$P=\{(\delta X)^2+(\delta Y)^2\}^{\frac{1}{2}}/\sin^2\phi,$$

on which the image-quality mainly depends, for each point (u, v) of the aperture-stop.

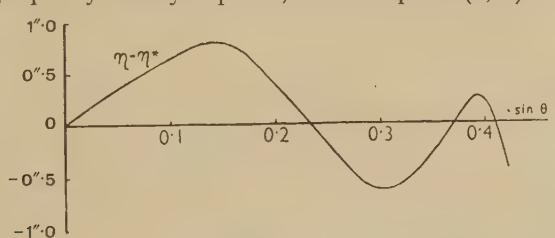


FIG. 8.

The error in $\delta X+i\delta Y$ caused by using η^* in place of η never amounts to more than a small fraction of its absolute value and can therefore be disregarded. Fig. 9 shows the distribution of values of P over the aperture-stop; only the first quadrant is displayed, since the complete figure is symmetrical about both axes.* The corresponding values

* Our thanks are due to Miss M. J. Littleton and Mr. E. Wolf for their help in the computing and drawing of Figs. 9, 10, 11 and 12.

FIG. 9.—*Monochromatic errors of f/1·2 meniscus Schmidt camera.*

FIG. 10.—*Monochromatic errors of f/1·2 Schmidt camera.*

for a classical Schmidt camera of the same focal ratio are shown in Fig. 10. On comparing the two figures, we see that at a focal ratio of $f/1{\cdot}2$ the monochromatic aberrations of the meniscus Schmidt are less than one-fifteenth of those of the classical Schmidt.

A good idea of the appearance of the off-axis images may be obtained by plotting in the X, Y plane the deviations $(\delta X, \delta Y)$ of the set of rays through the corner-points

$$u, v = 0, \quad \pm a, \quad \pm 2a, \quad \pm 3a, \ldots; \quad u^2 + v^2 \leqslant 1$$

of a lattice of small squares filling the entry-pupil. This gives a cluster of points whose density-distribution represents fairly accurately the intensity-distribution in the off-axis image, provided the aberrations reach a size at which diffraction effects play only a minor part. Fig. 11 shows the point-clusters obtained in this way for the meniscus Schmidt and the classical Schmidt at 6° off-axis.

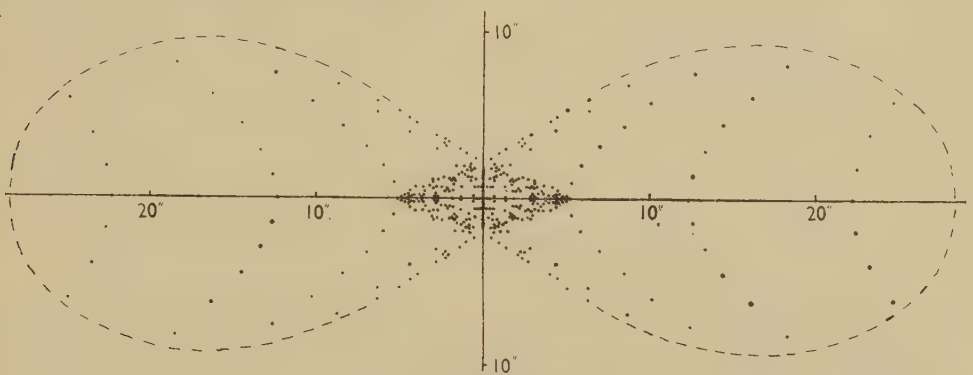
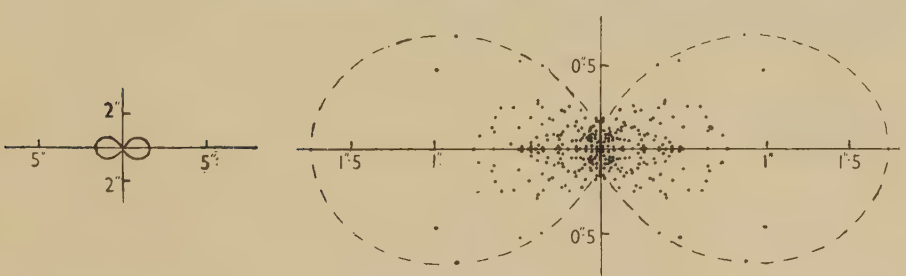


FIG. 11.

A. *f/1·2 Schmidt camera; monochromatic image 6° off-axis.*



B. *f/1·2 meniscus Schmidt camera; monochromatic image 6° off-axis, on the same scale as A.*
C. *The same, magnified ten diameters.*

It will be seen from Fig. 9 that, over a field 18° in diameter, the monochromatic error-spreads of the meniscus Schmidt are everywhere less than 8″ of arc and that 97 per cent. falls in a circle of diameter less than 3″·5 of arc. Since the remaining 3 per cent. of the light all comes from the extreme edge of the aperture, as we see from Fig. 9, it is worth while in some applications to place two extra stops, equal in diameter to the plate-aperture, at distances before and behind the aspheric surface equal to one-sixth of the plate-diameter. These stops reduce by about six per cent. the illumination at the edge of the field, and the result is a system, nearly free from edge-vignetting, whose monochromatic error-spreads are less than 3″·5 of arc and whose colour-spreads (C–F) are less than 8″ of arc over a curved field 18° in diameter.* The field-surface and its mounting

* Applied to the classical Schmidt, this device would involve much heavier edge-vignetting (about 25 per cent.) at the edge of the field, and so is hardly practicable, at any rate in astronomical applications.

cause a central obstruction which lowers the effective aperture from $f/1{\cdot}2$ to $f/1{\cdot}3$. Fig. 12, A, shows the point-cluster corresponding to the image at the edge of the field; account has been taken of the laterally displaced central obstruction as well as of the edge-vignetting. Fig. 12, B, shows the appearance of the effective aperture.

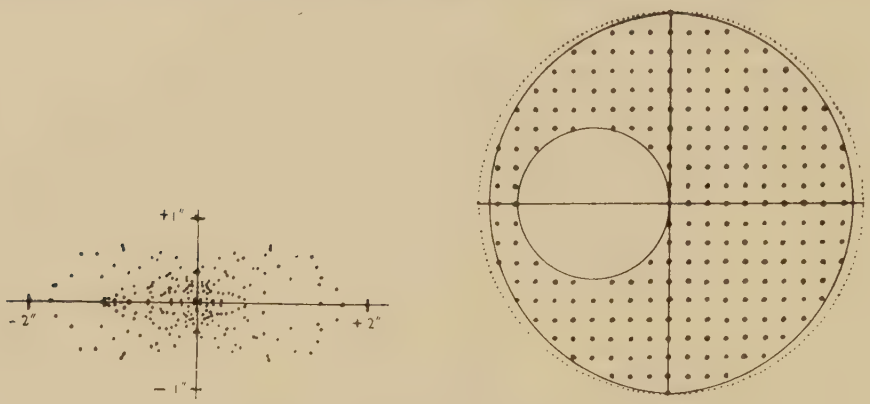


FIG. 12.

A. *$f/1{\cdot}2$ meniscus Schmidt with stops added; image 9° off-axis.*
B. *Effective aperture 9° off-axis.*

Lastly, if a wider field is desired we can, by shifting the extra stops to positions one-tenth of the plate diameter before and behind the aspheric surface, obtain a system, nearly free from edge-vignetting, whose monochromatic error-spreads are less than 10″ of arc and whose colour-spreads are less than 11″ of arc over a curved field 30° in diameter. The total error-spreads in this case are less than one-sixteenth of those of a classical Schmidt camera of the same field and aperture-ratio. The central obstruction is about $7\frac{1}{2}$ inches in diameter and lowers the effective aperture to $f/1{\cdot}5$.

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ON THE STRUCTURE OF DISK-SHAPED EXTRAGALACTIC NEBULÆ III. ON THE FORMATION OF MULTIPLE SYSTEMS

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Summary

The plan of the present paper is:

(i) To give a general picture of the formation of binary systems. It is supposed that the stars forming a binary condense in a disk-shaped nebula of the type discussed in Papers I and II (that is, a nebula in which the main contribution to the mass is due to interstellar hydrogen). The way in which stars in such a nebula may come to form binary systems is discussed in Section 2 where it is shown that stars are unlikely to form a binary unless they condensed from interstellar material that initially occupied two neighbouring regions of the nebula.

(ii) An independent line of investigation is followed in Sections 3 and 4. Firstly it is shown in Section 3 that the angular momentum of a binary can be determined within fairly narrow limits in terms of the initial masses of the components. Then, in Section 4, by a comparison of the observed angular momenta of binaries with the calculations of Section 3 an estimate is made for the initial masses of a selection of binary systems (Tables I and II). It is found that the initial masses so determined are consistently less, by a factor of order 10, than the observed masses of the components. This result is interpreted as strong evidence in favour of the theory that the components have increased significantly in mass due to accretion of interstellar hydrogen, during the time that has elapsed since the formation of the binary systems.

(iii) The work of Sections 2 and 4 is applied in Section 5 to a consideration of the general properties of binary systems with special reference to the observational data given by Aitken.*

(iv) In Section 6 the general picture of the formation of binaries is extended to cover the formation of triple and quadruple systems. It is shown that when the component stars in such systems undergo accretion of interstellar material at a sufficiently rapid rate, then

(*a*) almost all triple systems will consist of a relatively close pair together with a comparatively distant companion.

(*b*) About 5/7 of quadruple systems will consist of two close pairs separated by a comparatively large distance, whilst the other 2/7 of quadruple systems will consist of

(relatively close pair) + (comparatively distant companion) + (still more distant companion).

These considerations can be extended to systems containing more than four components.

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1. *Introduction.*—A theory of the formation of multiple systems must satisfy a number of well-known requirements:—

(i) About 50 per cent. of the stars in the neighbourhood of the Sun are members of

* R. G. Aitken, *The Binary Stars*, p. 203, 1935.

binary systems and about 10 per cent. of the stars are members of multiple systems containing more than two components.* The restriction of this statement to stars in the neighbourhood of the Sun arises from the fact that the properties of multiple systems can only be adequately observed if their distance from the Sun is not too great; for components of large luminosity (absolute magnitude of about -3) the limiting distance is about 1000 parsecs, whilst for components of small luminosity (absolute magnitude of about $+10$) the limiting distance is as little as 10 parsecs.

(ii) Both the period and the separation between the components are found to vary very widely from one binary system to another. Thus at the upper end of the scale binary systems are known † with separations of order 10^{17} cm., whilst spectroscopic binaries with the components practically in contact form the lower end of the scale, the separations in the latter case being of order 10^{11} cm. The periods show an even wider range than the separations, varying from about 300,000 years at the upper end of the scale down to periods of the order of a day at the lower end. It may be noted here that the methods that are used for observing binary systems introduce strong selection effects. Thus the spectroscopic method is usually only applicable to binaries with periods of less than fifty days, whilst the visual method is normally only applicable to binaries with periods lying between 10 and 1000 years. Accordingly the available observational data is necessarily confined in the main to these two ranges of period. It is usually regarded as reasonable to suppose that binaries occur with equal frequency in the ranges of period that are not accessible to observation.

(iii) The eccentricity in the orbit of a binary is strongly correlated with the separation of the components. Thus the eccentricity decreases very markedly with decreasing separation of the components. For spectroscopic binaries the eccentricity is always close to zero, whilst for visual binaries (separations usually of order 10 astronomical units) the eccentricity may be as high as 0·9 although 0·5 is a more normal value.

(iv) An interesting one-way correlation exists between mass and separation. Binaries with components of large mass (of the order of or greater than 5 times the solar mass $M_{\odot}$) are usually of small separation. The correlation does *not* hold, however, in the reverse direction, for it is by no means true that binaries of small separation are usually of large mass.

(v) The spacing of the components in multiple systems containing more than two components shows important regularities. For example, triple systems usually consist of a relatively close pair of stars together with a comparatively distant companion. Again quadruple systems are often found to consist of two relatively close pairs separated by a comparatively large distance. These properties of triple and quadruple systems provide an important check on any theory of binary stars that seeks to explain the requirements of (i), (ii), (iii) and (iv) described above. For it is evidently necessary that a complete theory of the formation and evolution of binary systems must also provide an explanation of the regularities observed in multiple systems containing more than two components.

An attempt has been made in previous work ‡ to give a theory capable of satisfying the requirements (ii) and (iii). It was assumed in this theory that

(*a*) Binaries are formed by collisions between stars, the separation at the time of formation being of order 10^{17} cm.

(*b*) The components of binary systems increase their masses very appreciably by accretion of interstellar hydrogen.

The important point in the theory is that it can be shown that the material added by

* G. P. Kuiper, *Ap. J.*, **95**, 201, 1942. A star means a single compact body. Thus a binary contains two stars.

† G. P. Kuiper, *loc. cit.*

‡ F. Hoyle and R. A. Lyttleton, *Proc. Camb. Phil. Soc.*, **35**, 592, 1939.

accretion must have negligible angular momentum about the centre of gravity of the binary. This has the effect of decreasing both the period and separation in accordance with the relations *

$$\left.\begin{array}{ll} \text{Period} & \propto (M_1+M_2)/M_1^3M_2^3, \\ \text{Separation} & \propto (M_1+M_2)/M_1^2M_2^2, \end{array}\right\} \quad (1)$$

where M_1, M_2, the masses of the components, are functions of time.

The theory described in the previous paragraph is further developed in the present paper where the previous assumptions (*a*) and (*b*) are examined. The important result emerges that a consideration of (*a*) leads naturally to a confirmation of (*b*). The chain of reasoning by which this is achieved is as follows: The discussion of the requirement (*a*) leads to the determination, within fairly narrow limits, of the angular momentum of a binary in terms of the masses of the components at the time of its formation. Thus it is possible to infer the initial masses of the components immediately from the observed angular momentum. A comparison of the initial mass, so determined, with the present observed masses gives the factor by which the components have increased in mass since the formation of the binary. This procedure, carried out for a large number of binaries, gives a factor of increase that in almost all cases lies between 5 and 30. This result establishes the requirement (*b*). The initial masses of the components turn out in almost all cases to lie between $M_\odot/50$ and $M_\odot$.

The consideration of the way in which binary systems form has other results in addition to those mentioned above. For the general picture of binary formation taken together with accretion of interstellar hydrogen is sufficient to satisfy the remaining requirements (i), (iv) and (v). Thus the further development of the former work * given in the present paper is able to explain the whole of the requirements laid down above; requirement (i) is considered in Section 2; requirement (ii) in Section 4; requirements (iii) and (iv) in Section 5; requirement (v) in Section 6.

Two of the conclusions established in the present paper have much wider implications than the immediate context in which they are given. Firstly, the result that the components of binary systems attain masses that are large compared with their initial masses, provides almost conclusive proof that accretion of interstellar hydrogen is a general process that takes place on the scale suggested in previous work.† Secondly, the result that the initial masses of the components of binary systems almost always lie between $M_\odot/50$ and $M_\odot$ has an important application to the problem of the luminosity function. For it provides direct confirmation of the main argument of Section 3 of Paper II, where it was shown that the masses of the stars at the time of their condensation must in general be less than $M_\odot$.

2. *The Formation of Binary Stars*

2.1. *Preliminary Remarks.*—In the present paper binary systems are regarded as being formed in encounters between stars. It is well known that when there are no external forces acting on an assembly of stars binary systems cannot be formed in two body encounters, since the stars in such encounters follow hyperbolic paths. Binary systems can, however, be formed in encounters where three or more stars are involved. The same situation arises when an external force such as the gravitational field of the nebula acts on the stars, provided the external force is constant in time. Thus the fact that permanently stable binary systems cannot be formed in two body encounters follows directly from the equations of motion which are unchanged in form when the sign of the time co-ordinate is reversed (this means that even if through the agency of the external force a binary system were temporarily formed in a two-body encounter, the external force must also be capable of breaking up the binary into single stars again). In the

* F. Hoyle and R. A. Lyttleton, *loc. cit.*

† F. Hoyle and R. A. Lyttleton, *M.N.*, 101, 227, 1941.

following discussion it will be supposed that over the short periods of time involved in the stellar encounters ($\simeq 10^6$ years) the gravitational field of the nebula can be regarded as independent of time.

Fortunately the general problem of three body encounters taking place in the gravitational field of the nebula is much simplified by the fact that the gravitational field of the nebula does not have any appreciable effect on the relative velocities of the stars *during* a "binary-forming" encounter, although it does have an important effect on the relative velocities before the encounter begins. This is due to the situation that during the encounter itself the distances between the stars turns out, as shown in the last paragraph of 2.3, to be small enough ($\simeq 10^{17}$ cm.) for the differential effects of the gravitational field of the nebula to be neglected (this is a further reason, in addition to that given in the previous paragraph, why the gravitational field of the nebula is unable to produce binaries in two body encounters). It is therefore permissible to divide the problem of binary formation into two parts. Firstly, to consider the question of stellar encounters omitting the gravitational field of the nebula, and secondly, to determine the relative velocities between the stars before an encounter begins by discussing the effect of the gravitational field of the nebula on the motions of the stars. These two questions are investigated separately in 2.2 and 2.3 respectively.

The two most important results obtained in the following discussion are:—

(*a*) That most stars must be members of binary systems provided the stars condense at distances apart of the order of, or less than, 10^{18} cm.

(*b*) That the separation between the components of a binary at the time of its formation must in general be greater than $S/10$, where S is the average interstellar distance.

The conclusion (*a*) is reversible and it is more convenient to present the argument in the inverted form. That is, to start by assuming that most stars are members of binary systems and to deduce the important inequality (15) which is both a necessary and sufficient condition for this to be the case. Then having obtained (15) it is shown in 2.3 that this condition can be satisfied provided the average interstellar distance is of order 10^{18} cm. The conclusion (*b*) appears as a by-product in this argument.

2.2. *Encounters between Stars.*—To begin with, consider an encounter between two stars of masses M and M'. Let v be the velocity of the star of mass M relative to the centre of gravity of the two stars before the encounter begins (that is, when the distance between the stars is large compared with their distance of closest approach), and let Mvp be the angular momentum of the star of mass M about the centre of gravity. Then it can be shown that *

$$(1-\sin\tfrac{1}{2}\psi)/\sin\tfrac{1}{2}\psi=(M+M')^2v^2\sigma/GM'^3, \tag{2}$$

where ψ is the angle of deflection of the orbit of the star in the encounter, σ is the distance of closest approach of the star of mass M to the centre of gravity, and G is the constant of gravitation. Moreover, the quantity σ is related to p by the equation

$$p^2=\sigma^2+2GM'^3\sigma/(M+M')^2v^2. \tag{3}$$

Now for the purpose of binary formation we are interested in those encounters in which the deflection ψ is considerable (that is, in encounters in which the gravitational potential between the two stars at the time of closest approach is comparable with the relative kinetic energy between the stars). For the sake of definiteness consider an encounter in which $\psi=\pi/2$. Then eliminating σ between (2) and (3) gives

$$p=GM'^3/(M+M')^2v^2, \tag{4}$$

the value of σ being

$$\sigma=(\sqrt{2}-1)GM'^3/(M+M')^2v^2. \tag{5}$$

* J. H. Jeans, *Astronomy and Cosmogony*, p. 309, Cambridge, 1936.

The equations (4) and (5) can now be applied to the question of triple encounters. To simplify the consideration of this problem it will be supposed that all stars are of mass M and move with a velocity v relative to any given star. The first step is to determine the probability of the given star experiencing an encounter of the type suited to binary formation. That is, the given star must have an encounter with a second star in which the distance of closest approach is of order

$$(\sqrt{2}-1)GM/4v^2, \tag{6}$$

and during this encounter a third star must approach the first and second stars also at a distance of the order given by (6). Under these circumstances, and provided a suitable configuration of the three stars occurs, the third star can remove an energy of order

$$GM^2/p = 4v^2M$$

from the system comprising the first and second stars, and this energy is of the same order as relative kinetic energy of the first and second stars before the encounter began. Thus although the total energy, gravitational + kinetic, of the system comprising the first and second stars would otherwise have been positive after the encounter, the removal of energy by the third star can under favourable circumstances lead to a system with negative total energy; that is, to the formation of a binary system (this is a brief statement of the well-known properties of triple encounters). It should be noted that the separation between the components of the binary at the time of its formation must in general be of the order given by (6).

Let P be the probability per unit time that the given star has an encounter with a second star in which the distance of closest approach is of the same order as, or less than, (6). Then the average time between such encounters is simply $1/P$. Now the duration of an encounter is of order

$$\sigma/v = (\sqrt{2}-1)GM/4v^3, \tag{7}$$

which is a fraction $(\sqrt{2}-1)GMP/4v^3$ of the average time between successive encounters. It follows therefore that

$$(\sqrt{2}-1)GMP/4v^3 \tag{8}$$

is the probability that a third star will approach the first and second stars at a distance of the order given by (6) during the encounter of the first and second stars. Thus the probability that a given star will experience a triple encounter of the required type is given by multiplying (8) by P. That is, the probability per unit time of a given star having a triple "binary-forming" encounter is of order

$$(\sqrt{2}-1)GMP^2/4v^3, \tag{9}$$

and the time between successive triple encounters is accordingly given by

$$4v^3/(\sqrt{2}-1)GMP^2.$$

The next step is to use the observed fact that a large fraction of stars are members of binary systems. This requires that on the average any given star should have at least one triple encounter of the type discussed in the previous paragraph during the age of the universe. Thus in this case it is necessary that

$$(\sqrt{2}-1)GMP^2T/4v^3 > 1, \tag{10}$$

where T is the age of the universe. This inequality leads to an important condition, as it is easy to determine the value of P. Thus the number of two body encounters in which

$$p \leqslant GM/4v^2 \tag{11}$$

experienced by a given star in time τ is

$$\pi\nu v\tau(GM/4v^2)^2, \tag{12}$$

where ν, the density of stars in space, is given in terms of the mean interstellar distance S by

$$4\pi S^3\nu/3 = 1. \tag{13}$$

It follows immediately that the average time interval between successive two body encounters satisfying (11) is $16v^3/\pi G^2M^2\nu$ so that P is given by

$$P = \pi G^2M^2\nu/16v^3. \tag{14}$$

Substituting this expression for P in (10) leads immediately to

$$v < [4.10^{-3}G^5M^5\nu^2T]^{1/9}. \tag{15}$$

This inequality for v is a necessary and sufficient condition for a large proportion of the stars to be members of binary systems. Before discussing the conditions under which (15) is satisfied, it is of interest to show from the results derived above that the separation of the components of a binary at the time of its formation cannot in general be less than about $S/10$. This follows immediately from the fact that the initial separation must in general be of the order of σ given by (5). Thus putting $M' = M$ in (5) and using (15) and (13) gives

$$[\text{separation in general}] > 3{\cdot}5.10^{-1}(GM)^{-1/9}T^{-2/9}S^{4/3}$$

or

$$\text{Separation}/S > 3{\cdot}5.10^{-1}(GM)^{-1/9}T^{-2/9}S^{1/3}. \tag{16}$$

The right-hand side of (16) is insensitive to the numerical values chosen for M, T and S. For the case $M = M_\odot$, $T = 10^{10}$ years $= 3.10^{17}$ sec., and $S = 10^{18}$ cm., the inequality (16) becomes

$$\text{Separation}/S > 5{\cdot}8.10^{-2}. \tag{17}$$

This establishes the required result that the initial separation cannot in general be much less than $S/10$. For $S = 10^{18}$ cm. the inequality (17) gives a lower limit of $5{\cdot}8.10^{16}$ cm. for the initial separation. This value is comparable with, but less than, the largest separations obtained by Kuiper in his survey of binary systems lying within about 10 parsecs from the Sun.*

2.3. *The Effect of the Gravitational Field of the Nebula.*—In the present subsection the conditions under which the inequality (15) is satisfied will be examined. That is, it is necessary to show in what circumstances a group of stars can be found that satisfy (15).

For the purpose of obtaining a numerical estimate for the right-hand side of (15) put $M = M_\odot$, $T = 3.10^{17}$ sec., and $S = 10^{18}$ cm. This gives $v < 1{\cdot}1.10^4$ cm. per sec., which requires so small a value of v that it is natural to suggest that the stars forming a binary system must condense in neighbouring regions of the nebula. Thus let two stars form at a distance b apart, where b is small compared with the thickness t of the disk of interstellar hydrogen. Then it can be seen from the discussion of Section 2 of Paper II that the two stars will in general have a component of relative velocity parallel to the plane of the nebula that is of order

$$b\cos\lambda\,|\,dV/dr\,|, \tag{18}$$

where λ is the angle between the displacement b and the plane of the nebula, and $V(r)$ is the rotational velocity of the interstellar material at distance r from the centre of the nebula. The analysis of stellar motions suggests that $|\,dV/dr\,|$ has a value of about

* G. P. Kuiper, *loc. cit.*

10^{-15} sec.$^{-1}$ in the region of the Galaxy within 1000 parsecs of the Sun ($|dV/dr| = 2.10^{-15}$ sec.$^{-1}$ was taken in Paper II as a typical value for the main body of a nebula). For this value of $|dV/dr|$ the relative velocity component given by (18) is less than $1{\cdot}1.10^4$ cm. per sec. provided

$$b \cos \lambda < 1{\cdot}1.10^{19} \text{ cm.} \tag{19}$$

Thus so far as the component of relative velocity parallel to the plane of the nebula is concerned, the condition (15) will be satisfied for all stars condensing within a region with dimensions of order ten times the average interstellar distance. It remains to show that the component perpendicular to the plane of the nebula will also satisfy (15) under these conditions. This question will now be considered.

Consider, to begin with, the condensation of a single star. Let r, ψ, z be cylindrical co-ordinates in an approximately axially symmetric nebula, the z-axis being perpendicular to the plane of the nebula with $z = 0$ on the central plane. Suppose that the star condenses at r_1, ψ_1, z_1 where $|z_1|$ is $>> b$ but $< t$. In addition to its orbital motion parallel to the plane of the nebula the star will also execute oscillations of amplitude $|z_1|$ in a direction perpendicular to the plane of the nebula. Thus while the material of the star was in a diffuse state it was supported against the component in this direction of the gravitational field of the nebula in accordance with equation (1) of Paper I, but after contraction to a compact stellar body no such equation of support is possible.

The equation determining the oscillations of the star is easily found. Write $\rho(r, \psi, z)$ for the density distribution of the interstellar hydrogen at the time of formation of the star, and let

$$\rho(r, \psi, z) = \rho(r, \psi, -z).$$

It is known from the discussion of Section 2 of Paper I that provided there are no large local fluctuations in the temperature T_H of the interstellar hydrogen, then

$$\frac{\partial \rho}{\partial z} >> \frac{\partial \rho}{\partial r} \quad \text{and} \quad \frac{\partial \rho}{\partial z} >> \frac{1}{r}\frac{\partial \rho}{\partial \psi} \tag{20}$$

and

$$\rho(r, \psi, z) = \rho(r, \psi, 0)e^{-q|z|/t}, \tag{21}$$

where q is a factor of order unity and t is given by equation (4) of Paper I. It can be shown from (20) and (21) that the component in the z-direction of the gravitational acceleration at the point r_1, ψ_1, z due to the interstellar hydrogen is

$$4\pi G t \rho(r_1, \psi_1, 0)(1 - e^{-q|z|/t})/q \tag{22}$$

and is directed towards the central plane of the nebula. If it is assumed that there are no large scale alterations in the distribution of the interstellar hydrogen, then the equation of oscillation of the star is evidently given by

$$\frac{d^2z}{d\tau^2} = -\frac{4\pi G t \rho(r_1, \psi_1, 0)(1 - e^{-q|z|/t})}{q} \tag{23}$$

when it is assumed that the main mass in the region r_1, ψ_1 of the nebula is due to the interstellar hydrogen. The latter assumption is almost certainly satisfied in the main body of a nebula, but it is possible that in the outer fringes the mass density νM due to the stars may be comparable with, or greater than, the density of the interstellar hydrogen. Thus since we are interested particularly in the region within 1000 parsecs of the Sun, which probably lies in the outer fringes of the Galaxy, it is possible that this assumption may not be applicable in the present case. If this should be so, an equation similar to (23) has to be constructed in which $\rho(r_1, \psi_1, 0)$ is replaced by the mass density of the stars and t by the thickness of the stellar distribution. The discussion of the properties of such an equation would follow along lines similar to the discussion given below.

Equation (23) has a first integral

$$\frac{dz}{d\tau} = \pm\left[\frac{8\pi G t\rho(r_1, \psi_1, 0)}{q}.\{|z_1| - |z| + (t/q)(e^{-q|z_1|/t} - e^{-q|z|/t})\}\right]^{\frac{1}{2}}, \tag{24}$$

the interpretation of the $\pm$ sign being that $dz/d\tau$ changes sign whenever it passes through a zero value (that is, whenever $|z| = |z_1|$). The period of oscillation of the star is

$$\frac{4}{\sqrt{(8\pi G t\rho(r_1, \psi_1, 0)/q)}}\int_0^{|z_1|} \frac{dz}{\{|z_1| - |z| + (t/q)(e^{-q|z_1|/t} - e^{-q|z|/t})\}^{\frac{1}{2}}}. \tag{25}$$

Consider now a second star condensing at the point $r_1 + \delta r$, $\psi_1 + \delta\psi$, $z_1 + b\sin\lambda$, where

$$\delta r^2 + r^2\delta\psi^2 = b^2\cos^2\lambda.$$

The second star will not in general condense at the same time as the first star. This means that although the distance between *the points of condensation* of the two stars is b, the distance between the stars themselves at the time of condensation of the second star will in general be of order $|z_1|$ which is $>> b$ (it being supposed that the second star condenses after the first). Although the stars are initially separated by a large distance of order $|z_1|$ it is easy to see that subsequently they will approach each other until their distance apart is of the order of, or less than, b. To see this consider the equations governing the motion of the second star. These equations are given by replacing r_1, ψ_1, $|z_1|$ in (23), (24) and (25) by $r_1 + \delta r$, $\psi_1 + \delta\psi$, $|z_1| + b\sin\lambda$. Now the co-ordinates $r_1 + \delta r$, $\psi_1 + \delta\psi$ appear only in the functions $\rho(r_1 + \delta r, \psi_1 + \delta\psi, z)$ and $t(r_1 + \delta r, \psi_1 + \delta\psi)$, and for δr and $r\delta\psi$ both of order b it is sufficient to write

$$\left.\begin{aligned}\rho(r_1 + \delta r, \psi_1 + \delta\psi, z) &= \rho(r_1, \psi_1, z)\\ t(r_1 + \delta r, \psi_1 + \delta\psi) &= t(r_1, \psi_1).\end{aligned}\right\} \tag{26}$$

Thus the equation for the velocity of the second star can be written to the first order in small quantities as

$$dz/d\tau = \pm\sqrt{(8\pi G t\rho(r_1, \psi_1, 0)/q)}\{|z_1| - |z| + b\sin\lambda(1 - e^{-q|z_1|/t}) + (t/q)(e^{-q|z_1|/t} - e^{-q|z|/t})\}^{\frac{1}{2}}, \tag{24'}$$

and the period of the second star is

$$\frac{4}{\sqrt{(8\pi G t\rho(r_1, \psi_1, 0)/q)}} \times \int_0^{|z_1| + b\sin\lambda} \frac{dz}{\{|z_1| - |z| + b\sin\lambda(1 - e^{-q|z_1|/t}) + (t/q)(e^{-q|z_1|/t} - e^{-q|z|/t})\}^{\frac{1}{2}}}. \tag{25'}$$

The periods (25) and (25′) are in general slightly different on account of the $b\sin\lambda$ term in (25′). This means that whatever the position of the first star at the time of condensation of the second, there must be a subsequent time when both stars have the same value, z' say, of the z co-ordinate. It is at this time that an encounter of the type discussed in 2.2 may occur. The component of relative velocity in the z-direction is then given by

$$\sqrt{(8\pi G t\rho(r_1, \psi_1, 0)/q)}[\{|z_1| - |z'| + (t/q)(e^{-q|z_1|/t} - e^{-q|z'|/t})\}^{\frac{1}{2}} - \{|z_1| - |z'| + b\sin\lambda(1 - e^{-q|z_1|/t}) + t/q(e^{-q|z_1|/t} - e^{-q|z'|/t})\}^{\frac{1}{2}}]. \tag{27}$$

Apart from in the special case where z' is such that $1 - |z'|/|z_1| << |$ the expression (27) can be written in the form *

* When $|z'| = |z_1|$ the relative velocity (27) becomes

$$-\{8\pi G t\, b\sin\lambda\rho(r_1, \psi_1, 0)(1 - e^{-q|z_1|/t})/q\}^{\frac{1}{2}},$$

which is greater in magnitude than (28) by a factor of order $\{|z_1|/b\}^{\frac{1}{2}}$. Thus in seeking the conditions leading to the smallest relative velocity we may omit this case.

$$-\frac{b \sin\lambda\sqrt{(2\pi G t\rho(r_1, \psi_1, 0)/q)}(1-e^{-q|z_1|/t})}{\{|z_1|-|z'|+(t/q)(e^{-q|z_1|/t}-e^{-q|z'|/t})\}^{\frac{1}{2}}}. \tag{28}$$

Thus since

$$\sqrt{(t/q)}(1-e^{-q|z_1|/t})/\{|z_1|-|z'|+t/q(e^{-q|z_1|/t}-e^{-q|z'|}\}^{\frac{1}{2}}$$

is of order unity (the special case $1-|z'|/|z_1| << |$ being excluded) it follows that the relative velocity (28) must have a magnitude of order $\sqrt{\{2\pi G\rho(r_1, \psi_1, 0)\}}b \sin\lambda$. It follows therefore that if the inequality (15) is to be satisfied by this relative velocity then we must have

$$b \sin\lambda << \text{a quantity of order } (4.10^{-3}G^5M^5\nu^2T)^{1/9}/\sqrt{(2\pi G\rho(r_1, \psi_1, 0))}. \tag{29}$$

The right-hand side of (29) does not depend strongly on $\rho(r_1, \psi_1, 0)$. Thus although no definite numerical value is available for $\rho(r_1, \psi_1, 0)$ we can nevertheless obtain a fairly reliable estimate for the condition on $b \sin\lambda$ by choosing what seems to be a reasonable value for $\rho(r_1, \psi_1, 0)$. For example * put $\rho(r_1, \psi_1, 0) = 10^{-23}$ gm. per cm.^{3} for the region of the Galaxy within 1000 parsecs from the Sun and again take

$$M = M_\odot, \qquad T = 3.10^{17} \text{ sec.}, \qquad S = 10^{18} \text{ cm.}$$

This gives

$$b \sin\lambda < 5.10^{18} \text{ cm.} \tag{30}$$

The condition (19) ensures that the component of relative velocity parallel to the plane of the nebula shall satisfy (15), whilst (30) is necessary in order that encounters occur in which the component of relative velocity perpendicular to the plane of the nebula also satisfies (15). These two conditions taken together show that all stars condensing within a distance of about 10^{19} cm. from the point of condensation of any given star will have encounters with the given star in which the total relative velocity satisfies (15). Accordingly there must be about $(10^{19}/S)^3$ stars available for suitable "binary-forming" encounters. Provided S is of the order of, or less than, 10^{18} cm. this number is sufficiently numerous (about 1000) to satisfy the requirement of 2.2; that there is a group of stars with velocities relative to the given star satisfying (15). On the other hand, for S of order 10^{19} cm. the number of stars available is of order unity, and this could hardly be regarded as an adequate "group of stars". It is therefore at this stage of the argument that the assumption of S of the order of, or less than, 10^{18} cm. must be introduced.

We may now recapitulate the sequence of arguments given in 2.2 and 2.3. Starting with the observed fact that a high proportion of the stars in the solar neighbourhood are members of binary systems it was shown in 2.2 that there must be a group of stars (with star density ν of order 10^{-55} per cm.^{3}) moving relative to any given star with velocities satisfying (15). The conditions under which this requirement is satisfied are examined in 2.3, where it is shown that a suitable group of stars must exist provided S is of the order of, or less than, 10^{18} cm. As stated in 2.1 this argument can be reversed. Thus starting from the assumption that stars condense at an average distance of order 10^{18} cm. apart, it is possible to show exactly as in 2.3 that a group of about 1000 stars condensing within a distance of about 10^{19} cm. from the point of condensation of any given star may have encounters with the given star in which the relative velocity satisfies (15). The argument of 2.2 can then be inverted. Instead of starting from the assumption that most stars are members of binary systems and working to the result that there

* The value of $\rho(r_1, \psi_1, 0)$ is very much less for this region of the Galaxy than in the main body of M31 or M33. This is due to the "mass per unit area of the nebular disk" being about 10^{-2} gm. in the neighbourhood of the Sun as against 1/5 gm. for the main body of M31 or M33. For 100° abs. $< T_H <$ 1000° abs. the value of $\rho(r_1, \psi_1, 0)$ probably lies between 10^{-22} gm. per cm.^{3} and 10^{-23} gm. per cm.^{3}, whilst for $T_H = 10{,}000°$ abs. the value of $\rho(r_1, \psi_1, 0)$ is probably of order 10^{-24} gm. per cm.^{3} (see Section 3).

must be a group of stars moving relative to any given star with velocities satisfying (15), we can work from the knowledge that such a group of stars exists to the result that a high proportion of stars must be members of binary systems. Thus the reverse of the arguments of 2.2 and 2.3 shows that provided stars condense at a distance of order 10^{18} cm. apart, then in a period of 10^9 years a high proportion of stars will have become members of binary systems.

The assumption stated in the second paragraph of 2.1 is easily shown to be satisfied. This assumption was that the action of the gravitational field of the nebula can be neglected in discussing "binary-forming" encounters between the stars. From (17) it follows that the distance between the stars in such encounters must be of order 10^{17} cm. On the other hand, the arguments leading to the inequalities (19) and (30) show that the differential effect of the gravitational field of the nebula will not introduce a relative velocity between stars as large as 10^4 cm. per sec. unless their distance apart is of order 10^{19} cm. Thus since the differential effect of the gravitational field of the nebula depends linearly on the distance b it follows that the variation in relative velocity during a "binary-forming" encounter, arising from the action of the gravitational field of the nebula, cannot exceed the negligible value of 10^2 cm. per sec. This establishes the required result.

3. *The Mean Interstellar Distance.*—It was shown above that provided the average interstellar distance is of the order of, or less than, 10^{18} cm. then a high proportion of stars must be members of binary systems. This gives an important condition on the average interstellar distance that must be satisfied in order to meet the first of the five items given in the Introduction. It is accordingly of interest to discuss the factors determining the mean interstellar distance.

Consider the condensation of stars in a volume V of the nebula. For simplicity let the interstellar material in this region be of uniform density ρ. If a fraction f of the mass of the interstellar material in this region is condensed into stars of mass M then the total mass of the stars in V is $fV\rho$ and the number of stars is $fV\rho/M$. It follows immediately that the star density ν is $f\rho/M$ and that the average interstellar distance is given by

$$S=\{3M/(4\pi f\rho)\}^{\frac{1}{3}}. \tag{31}$$

It is shown below in Section 4 that $M \simeq M_\odot/10$ may be regarded as a typical value for the mass of a star at the time of its condensation (see also Section 3 of Paper II). Using this value of M in (31) gives the following values of S corresponding to a selection of values of $f\rho$:—

$f\rho$	10^{-22} gm. per cm.^{3}	10^{-23} gm. per cm.^{3}	10^{-24} gm. per cm.^{3}
S	$7{\cdot}8.10^{17}$ cm.	$1{\cdot}7.10^{18}$ cm.	$3{\cdot}6.10^{18}$ cm.

Thus for $f\rho$ of the order of, or greater than, 10^{-23} gm. per cm.^{3} the value of S is close to 10^{18} cm., and even for $f\rho=10^{-24}$ gm. per cm.^{3} the value of S is not unduly large (the group of stars discussed towards the end of 2.3 would still have about 30 members). It may be noted that the value of S in the latter case corresponds fairly closely to the observed average interstellar distance for the region of the Galaxy lying within 10 parsecs from the Sun.*

It is useful to notice in connection with the above remarks, that the value of ρ can be obtained from equation (4) of Paper I which gives

$$\rho=\pi\mu G(\rho t)^2/9\Re T_H, \tag{32}$$

provided T_H and ρt the "mass per unit area of the nebular disk" are both known. The value of ρt has been estimated as being of order 10^{-2} (from an analysis of the peculiar velocities of the stars in a direction perpendicular to the Galactic plane), whilst the

* G. P. Kuiper, *loc. cit.* This may or may not be a typical value.

discussion of Section 2 of Paper II suggests that T_H is of the order of, or less than, 100° abs. in regions where stars are condensing. The value of ρ given by (32) corresponding to $\rho t = 10^{-2}$, $T_H = 100°$ abs., $\mu = 1$ is $2{\cdot}8.10^{-22}$ gm. per cm.^{3}. Thus in this case S will be close to 10^{18} cm. even if the fraction f is as small as 1/10. Even if $T_H = 5000°$ abs., $\rho t = 10^{-2}$ gm., $\mu = 1$, the value of ρ is $5{\cdot}6.10^{-24}$ gm. per cm.^{3} and $f\rho$ may be equal to 10^{-24} gm. per cm.^{3} without requiring f to be greater than 1/5·6. This shows that a condition $S < 3{\cdot}6.10^{18}$ cm. can be satisfied without imposing any serious restriction either on T_H or on f.

The above discussion together with the inequality (17) may be used to determine a lower limit for the angular momentum possessed by a binary system. Let a be the semi-major axis and e the eccentricity of the binary at the time of its formation. The angular momentum at this time is

$$M_1{}^i M_2{}^i \{aG(1-e^2)/(M_1{}^i + M_2{}^i)\}^{\frac{1}{2}}, \tag{33}$$

where $M_1{}^i$, $M_2{}^i$ are the initial masses of the components at the time of formation of the binary. Subsequent evolution of the binary arising from accretion of interstellar hydrogen will not alter this angular momentum since the hydrogen is added with negligible angular momentum. Thus the angular momentum possessed by a binary at any stage of its evolution is known in terms of $M_1{}^i$, $M_2{}^i$, a and e.

It is easy to determine a lower limit for (33). Consider firstly the dependence on a. It follows from (17) that

$$a > 5{\cdot}8.10^{-2} S.$$

Thus since the above discussion shows that S cannot be appreciably less than 10^{18} cm., it follows that

$$a > 5{\cdot}8.10^{16} \text{ cm.} \tag{34}$$

Substituting (34) in (33) gives

$$[\text{Angular momentum of binary}] > M_1{}^i M_2{}^i \{5{\cdot}8.10^{16} G(1-e^2)/(M_1{}^i + M_2{}^i)\}^{\frac{1}{2}}. \tag{35}$$

The next step is to determine the distribution of eccentricity among binary systems at *the time of their formation.* For this purpose we may use Jeans's analysis of the distribution of eccentricities for the case where encounters of the binaries with outside stars have been sufficiently numerous for statistical equilibrium to be reached.* This is a permissible application of Jeans's work provided the separation of the binary remains comparable with the average interstellar distance, for in this case neighbouring stars will constantly exercise an appreciable perturbation on the binary orbits (as shown below in Section 5, the distribution of eccentricities becomes markedly changed if accretion reduces the separation to values that are small compared with the mean interstellar distance). Jeans's expression for the equilibrium distribution shows that the fraction of binary systems with eccentricities lying between e and $e + de$ is proportional to $e.de$, the factor of proportionality being independent of e. Thus half of all binaries at the time of their formation will have eccentricities less than $1/\sqrt{2}$, from which it follows that half of all binaries must have angular momenta greater than

$$M_1{}^i M_2{}^i \{2{\cdot}9.10^{16} G/(M_1{}^i + M_2{}^i)\}^{\frac{1}{2}} = 4{\cdot}0.10^{54} M_1{}^i M_2{}^i M_\odot{}^{-\frac{3}{2}} / \sqrt{(M_1{}^i + M_2{}^i)}, \tag{36}$$

which depends only on the initial masses $M_1{}^i$, $M_2{}^i$ of the components. Furthermore, of the 50 per cent. of binaries with eccentricities $> 1/\sqrt{2}$ only a small fraction will have angular momenta appreciably less than (36). Thus for a star to have angular momentum less than

$$10^{54} M_1{}^i M_2{}^i M_\odot{}^{-\frac{3}{2}} / \sqrt{(M_1{}^i + M_2{}^i)}$$

it is necessary that the eccentricity be greater than $\sqrt{31/32}$, which according to Jeans's

* Jeans, *loc. cit.*, p. 305.

distribution is satisfied by only about 3 per cent. of binaries. It follows therefore that the great majority of binaries will have angular momenta that are not appreciably less then (36).

4. *Comparison with Observational Data.*—A selection of spectroscopic and visual binaries are given in Tables I and II. It should be noted that the apparent correlation between mass and angular momentum in Table I probably arises from a strong selection effect, for as seen from this table spectroscopic eclipsing binaries are most easily observed when the separation of the components lies between 10 and 20 solar radii. Moreover, a comparison with Table II provides further evidence that this correlation is not genuine.

The first point that stands out in comparing these binaries with the theory developed above is that the observed separations are very much less than the value given by (17) for the lower limit to the separation of the binary at the time of its formation. This shows that the separations and periods have both decreased by large factors since the time of formation of the binaries. As a time scale of 10^{10} years is much too short for encounters between the stars to be important in producing such changes of separation and period it is natural to suggest that the observed binaries in Tables I and II have evolved through accretion of interstellar material; the separations and periods changing in accordance with (1).

The same conclusion is reached by comparing the observed angular momenta of the binaries in Tables I and II with the values obtained by substituting the present masses of the components in (36). It is seen that the calculated angular momenta turn out to be greater than the observed values by an average factor of about 500 for the spectroscopic binaries and about 20 for the visual binaries. Thus unless we are prepared to assume that some external agency has removed a very large proportion of the initial angular momentum (amounting to about 99·8 per cent. for the spectroscopic binaries and about 95 per cent. for the visual binaries) it is necessary to suppose that the present masses of the components are larger than the initial masses by a very appreciable factor. In order to give agreement between the observed value of the angular momentum and the value calculated from (36) it is necessary for the masses of the components to have increased by a factor that in general lies between about 5 and 30. For example, the initial masses of the components of 29 CMaj must have been close to $M_\odot$, which requires a subsequent increase in mass by a factor of about 30. On the other hand, for τ Cygn the initial masses of the components must have been about $M_\odot/4$, which requires a subsequent increase by a factor of about 6. It is seen that the initial masses vary from one binary system to another. Thus taking the cases of 29 CMaj and S Antl, the initial masses of the components of 29 CMaj must have been about $M_\odot$, whilst the initial masses for S Antl must have been about $M_\odot/50$. These examples are extreme cases, the general run of the initial masses being of order $M_\odot/10$.

The above discussion provides strong evidence in favour of the following conclusions:—

(i) That the masses of the components of binary systems increase due to accretion of interstellar hydrogen.

(ii) That the initial masses of the components of a binary at the time of its formation are in general of order $M_\odot/10$.

Although these conclusions apply specifically to the particular group of stars at distances less than about 1000 parsecs from the Sun, it is reasonable to suppose that they have wider implications. Thus (i) provides strong confirmation of the general theory of accretion by stars, which is of importance

(*a*) In connection with the problem of the origin of the bright stars in a nebula.

(*b*) In affecting the axial rotation of the stars.

Thus it has been shown in Section 2 of Paper II that the stars may be expected to have very high axial rotations at the time of their condensation. If material is subsequently accreted with negligible angular momentum about the axis of rotation of the star then the rotational velocity of the star will be slowed down to a marked extent. Thus if M is the present mass of the star and M^c was the mass at the time of condensation, then the rotational velocity would be slowed down by a factor M^c/M if the star remained of the same radius throughout the accretion. But the radius of the star will increase as the mass of the star increases, and this reduces the rotational velocity still further. For main series stars, in which energy is generated by the carbon-nitrogen cycle, the radius is known to be approximately proportional to $M^{\frac{2}{3}}$, from which it can be seen that the rotational velocity at any stage is proportional to $(M^c/M)^{\frac{1}{3}}$. This means that the rotational velocity is slowed down by a factor of 46 when the mass increases by a factor of 10. Thus an increase in mass from $M_\odot/10$ to $M_\odot$ will slow an initial rotational velocity of order 5.10^7 cm. per sec. down to a velocity of order 10^6 cm. per sec.

The conclusion (2) gives direct confirmation of the arguments of Section 3 of Paper II, and is accordingly of considerable importance since it provides support for the suggestions made concerning the problem of the luminosity function.

5. *Some Statistical Properties of Binary Systems.*—In this section the items (3) and (4) given in the Introduction will be briefly discussed.

The marked correlation between separation and eccentricity in binary systems is shown by the examples given in Tables I and II. The eccentricities of the visual binaries are already systematically less than the *e.de* distribution used in Section 3. Thus it is seen from Table II that the eccentricity in visual binaries averages about 0·4, although values as high as 0·9 occur. The spectroscopic binaries have eccentricities very near zero, the largest value given in Table I being about 0·2.

These data, together with equations (1), which show that the separation decreases very markedly with accretion of interstellar material, suggests that accretion may also be responsible for the correlation between separation and eccentricity. Thus since on the one hand the separation decreases due to accretion, and on the other hand the eccentricity decreases with the separation, it is natural to seek a theory in which the decrease of eccentricity arises from the effect of accretion.

The increase in the masses of the components of a binary does not in itself lead to the required change in eccentricity. This can be seen from Jeans's analysis of the reverse problem *, in which the components lose mass by radiation. In this case the separation and period alter but the eccentricity of the orbit remains unchanged to the first order in small quantities. Thus unless the effect of accretion introduces a force acting on the components in addition to their mutual gravitational attraction there will be no important change of eccentricity (this follows, since unless such a force exists the equations determining the evolution of the binary orbit can be obtained by reversing the sign of the time co-ordinate in Jeans's equations). The existence of such a force has already been discussed in a previous paper †, where the problem of the motion of a star through a uniform gas was considered. It was shown that the star experiences a retardation which reduces the velocity relative to the gas. The time required for the velocity to be halved is of the same order as the time required for accretion to double the mass of the star. Thus the retarding drag of the gas produces important dynamical effects in a time of the same order as the time required for accretion to give an important change in mass. These results may be applied to the problem of the eccentricities of the binary orbits. The problem is essentially similar to that considered by Jeffreys in connection with the planetary orbits.‡ The drag of the gaseous accreted material on the components acts as a damping agency that rounds up the binary orbits. The time required for

* Jeans, *loc. cit.*, p. 300.

† H. Bondi and F. Hoyle, *M.N.*, **104**, 273, 1944.

‡ H. Jeffreys, *The Earth*, p. 58, Cambridge, 1929.

TABLE I

The data are taken from H. N. Russell and C. E. Moore, *The Masses of the Stars*, Chicago, 1940.
M_1 = mass of more massive component.
M_2 = mass of less massive component.
e = eccentricity of the orbit.

Star	Period (Days)	$M_1/M_\odot$	$M_2/M_\odot$	e	Angular Momentum	Major Axis (Solar Radii)
29 CMaj	4·39	44·5	33·0	0·16	$6{\cdot}9.10^{54}$	48·1
Y Cygn	3·00	17·6	17·4	0·13	$1{\cdot}7.10^{54}$	28·7
V Pupp	1·45	20·7	15·2	0·08	$1{\cdot}4.10^{54}$	17·8
AH Ceph	1·77	14·9	12·8		$9{\cdot}3.10^{53}$	18·5
σ Aqil	1·95	6·25	5·20	0·00	$2{\cdot}2.10^{53}$	14·7
TT Auri	1·33	6·65	5·30	0·00	$2{\cdot}1.10^{53}$	12·2
AG Pers	2·03	5·16	4·60		$1{\cdot}7.10^{53}$	14·2
U Ophi	1·68	5·30	4·60	0·00	$1{\cdot}6.10^{53}$	12·7
u Herc	2·05	7·30	2·80	0·05	$1{\cdot}4.10^{53}$	14·6
U Cor B	3·45	6·50	2·35	0·00	$1{\cdot}4.10^{53}$	20·1
U Sgte	3·38	6·65	2·05	0·03	$1{\cdot}2.10^{53}$	19·5
Z Vulp	2·45	5·30	2·36	0·00	$1{\cdot}1.10^{53}$	15·1
WW Drac	4·63	3·50	2·46		$1{\cdot}0.10^{53}$	21·1
AR Auri	4·13	2·40	2·35		$6{\cdot}7.10^{52}$	18·2
MR Cygn	1·68	3·04	2·63		$6{\cdot}7.10^{52}$	10·7
β Auri	3·96	2·35	2·35	0·00	$6{\cdot}5.10^{52}$	17·7
WW Auri	2·52	2·20	1·90	0·00	$4{\cdot}5.10^{52}$	12·5
RS CVen	4·80	1·86	1·73	0·00	$4{\cdot}3.10^{52}$	18·2
RX Herc	1·78	2·10	1·83	0·00	$3{\cdot}7.10^{52}$	9·8
TX Herc	2·06	2·05	1·76	0·00	$3{\cdot}7.10^{52}$	10·7
TX UMaj	3·06	3·05	0·88	0·02	$3{\cdot}1.10^{52}$	14·0
CM Lacr	1·60	2·00	1·48		$2{\cdot}9.10^{52}$	8·7
RT Lacr	5·07	1·00	1·90	0·00	$2{\cdot}9.10^{52}$	17·9
Z Herc	3·99	1·48	1·20	0·00	$2{\cdot}7.10^{52}$	15·0
WZ Ophi	4·18	1·40	1·33		$2{\cdot}7.10^{52}$	15·5
AR Lacr	1·98	1·44	1·44		$2{\cdot}3.10^{52}$	9·4
TV Cass	1·81	1·68	1·00	0·00	$2{\cdot}3.10^{52}$	8·6
GO Cygn	0·72	1·56	1·33		$1{\cdot}6.10^{52}$	4·77
YY Gemi	0·81	0·63	0·57		$3{\cdot}9.10^{51}$	3·9
S Antl	0·65	0·75	0·42	0·00	$3{\cdot}1.10^{51}$	3·29
W UMaj	0·33	0·73	0·53	0·00	$3{\cdot}1.10^{51}$	2·18

TABLE II

The data given in this table have been taken partly from H. N. Russell and C. E. Moore, *The Masses of the Stars*, and partly from R. G. Aitken, *The Binary Stars*. The major axis and the masses of the components are sometimes taken from different sources and this may lead to slight inconsistencies with the values given for the period.

Star	Period (Years)	$M_1/M_\odot$	$M_2/M_\odot$	e	Angular Momentum	Major Axis (Astronomical Units)
τ Cygn	49·16	1·86	1·25	0·22	$7{\cdot}0.10^{53}$	37·6
α CMaj	49·94	2·65	1·03	0·588	$6{\cdot}5.10^{53}$	40·6
ζ Canc AB	59·6	1·45	1·25	0·314	$6{\cdot}0.10^{53}$	41·2
γ Virg	177·75	1·43	1·43	0·859	$5{\cdot}1.10^{53}$	85·0
Σ 2173	46·0	1·18	1·15	0·18	$4{\cdot}6.10^{53}$	35·4
α Cent	80·09	1·10	0·93	0·521	$3{\cdot}7.10^{53}$	46·5
β 648	57·0	1·27	0·68	0·20	$3{\cdot}2.10^{53}$	34·4
ζ Herc	34·42	1·58	0·68	0·455	$2{\cdot}9.10^{53}$	25·5
70 Ophi	87·71	0·93	0·76	0·499	$2{\cdot}8.10^{53}$	45·4
η Cass	478·7	1·01	0·32	0·495	$2{\cdot}5.10^{53}$	132·4
β 395	25·0	0·90	0·86	0·171	$2{\cdot}2.10^{53}$	19·1
δ Equl	5·70	1·25	1·13	0·39	$2{\cdot}2.10^{53}$	8·2
99 Herc	56·0	1·37	0·59	0·787	$2{\cdot}0.10^{53}$	38·8
85 Pegs	26·3	0·89	0·73	0·46	$1{\cdot}7.10^{53}$	18·2
Kr 60	44·52	0·33	0·19	0·41	$3{\cdot}0.10^{52}$	18·4

the process to be effective is comparable with the time required for the components to increase appreciably in mass; that is, with the time required for an appreciable reduction in the separation of the components.

Consider now item (4) given in the Introduction. It is easy to show that there can be no general correlation between the masses and separations of binary systems. From equations (1) it follows that

$$\frac{\text{Present separation of Binary}}{\text{Initial separation of Binary at time of formation}} = \frac{M_1 + M_2}{M_1{}^i + M_2{}^i}\left(\frac{M_1{}^i M_2{}^i}{M_1 M_2}\right)^2. \tag{37}$$

Take two binaries α and β and let

$$\left.\begin{aligned} &M_1(\alpha) = M_2(\alpha) = M_1(\beta) = M_2(\beta), \\ &M_1{}^i(\alpha) = M_2{}^i(\alpha) = M^i(\alpha), \\ &M_1{}^i(\beta) = M_2{}^i(\beta) = M^i(\beta). \end{aligned}\right\} \tag{38}$$

That is, suppose that the present masses of the four components in α and β are all equal, but let the initial mass of α be different from the initial mass of β. This means that the two binaries must have different accretion rates. From (37) it is seen that

$$\frac{\text{Present separation of } \alpha}{\text{Present separation of } \beta} = \frac{\text{Initial separation of } \alpha}{\text{Initial separation of } \beta}\left[\frac{M^i(\alpha)}{M^i(\beta)}\right]^3. \tag{39}$$

Equation (39) shows that although the present masses of α and β are equal, the present separations can be widely different. This follows immediately from the discussion of Section 4, which showed that the initial mass of the components of a binary may lie between $M_\odot/50$ and $M_\odot$. Thus $[M^i(\alpha)/M^i(\beta)]^3$ may introduce a factor of 1000 or more into the right-hand side of (39). The ratio of the initial separations on the other hand can at most introduce a factor of about 10. This follows since, firstly, the initial separations must both be less than the mean interstellar distance S, and secondly, it is seen from (17) that both initial separations must be greater than $5{\cdot}8.10^{-2}S$.

Although for the reasons given in the previous paragraph no general correlation between mass and separation is possible, it is to be expected that binary systems of large mass (such as 29 CMaj) will in general be spectroscopic binaries rather than wide visual pairs. This is because large mass ($M >> M_\odot$) indicates a large increase of mass by accretion, since according to the discussion of Section 4 the initial masses of the components at the time of formation of the binaries do not appreciably exceed $M_\odot$.

For main series stars a correlation between large mass and small separation (the correlation only goes one way; large mass means small separation, but small separation does not necessarily imply large mass) is equivalent to a correlation between spectral type and period. That is, the earlier the spectral type the shorter the period. A correlation of this type has been found by Aitken *, who has shown that if P_B, P_A, P_F, etc. are defined by the condition that 50 per cent. of stars of class B have periods of less than P_B, 50 per cent. of class A have periods of less than P_A, etc. . . ., then the sequence P_B, P_A, P_F, . . . is an ascending sequence. This is exactly the type of correlation to be expected from the present theory. Aitken's data give the following values $P_B \simeq 5$ days, $P_A \simeq 10$ days, $P_F \simeq 20$ days, $P_G \simeq 500$ days, $P_K > 500$ days, $P_M > 500$ days.

6. *The Formation of Triple and Quadruple Systems.*—There are two ways in which triple systems can be formed in an assembly containing single stars and binary systems:—

(*a*) In an encounter between four stars. One of the four stars must remove sufficient energy to give a negative value for the total energy (gravitational + kinetic) of the remaining three stars.

(*b*) In an encounter involving a single star, a binary, and a third body (which may be either a single star or a binary).

* Aitken, *loc. cit.*, p. 211.

In this case the third body must remove sufficient energy to leave a negative total energy for the system comprising the single star and the binary.

It follows immediately from the discussion of 2.2 that

$$[\text{Probability of } (a)] \simeq [\text{Probability of binary formation}] . [\text{The expression (8)}], \quad (40)$$

where the expression (8) gives the probability of a fourth star being present in an encounter of three single stars. On the other hand the probability of forming a triple system by process (*b*) is given by

$$[\text{Probability of } (b)] \simeq [\text{Probability of binary formation}] \cdot \frac{\text{Space density of binaries}}{\text{Space density of single stars}}. \quad (41)$$

Now, as was seen in 2.2, the expression (8) is $<< 1$, while on the other hand the space density of binary systems is about $\frac{1}{2}$ the space density of single stars. Accordingly (41) is much larger than (40). It follows therefore that almost all triple systems are formed by process (*b*).

The above remarks can be generalised. Thus the probability of forming a multiple system with any given number of components is highest for encounters involving only three bodies. For example, the probability of forming a quadruple system either

(i) by a three body encounter involving [binary + binary + third body],

(ii) by a three body encounter involving [single star + triple system + third body],

is far higher than in an encounter involving five single stars, or in an encounter involving three single stars and a binary. It follows, therefore, that in discussing the general problem of the formation of multiple systems it is sufficient to consider only three body encounters.

Before proceeding to the determination of the relative proportions of single stars, binaries, triple systems, etc., it is of interest to deduce important results concerning the spacing of the components in triple and quadruple systems. These considerations provide an explanation of the item (5) of the Introduction. The required results follow almost immediately from the discussion of the previous two paragraphs. It has been seen that almost all triple systems must be formed from the association of a single star and a binary (a third body being present during the encounter). The conditions governing the initial separation between the single star and the binary are exactly the same as those leading to the inequality (17). Thus the separation between the single star and the binary at the time of formation of a triple system cannot be appreciably less than 10^{17} cm. Now although the separation of the components in the binary were also of order 10^{17} cm. at the time of formation of the binary, the separation of these components may be very much less at the time of formation of the triple system. For this to be the case it is only necessary that the binary system shall undergo appreciable accretion during the time interval between the formation of the binary and the formation of the triple system. The average value of this time interval, which may be of the same order as the ages of the stars themselves, can be regarded as sufficient for the binary to evolve from a separation of about 10^{17} cm. into a comparatively close pair. If this is the case then the triple system will consist of a comparatively close pair together with a relatively distant companion. This result is in accordance with the requirement set out in item (5) in the Introduction.* Similar considerations can also be applied to the spacing of the components in quadruple systems. Thus the quadruple systems formed by process (i) will consist of two comparatively close pairs separated by a relatively large distance. The quadruple systems formed by process (ii), on the other hand, will consist of

(Comparatively close pair)+(relatively distant companion)+(still more distant companion).

The discussion given below suggests that quadruple systems formed by process (i) are about two and a half times as numerous as the quadruple systems formed by process (ii).

* See also Aitken, *loc. cit.*, p. 274.

To conclude the present paper we shall consider the relative proportions of single stars, binaries, triple systems, etc. that will occur in an assembly of stars. Let the assembly contain N stars where $N >> 1$ and suppose that all stars are single at time $\tau = 0$. Write $n_1(\tau)$ for the number of single stars at time τ; $n_2(\tau)$ for the number of binaries; $n_3(\tau)$ for the number of triple systems, etc. For simplicity assume that no stars condense from the interstellar material during the time τ. Then we have

$$\sum_{p=1}^{\infty} p n_p(\tau) = N. \tag{42}$$

The probability at time τ of an encounter between a system containing p components and a system containing $r-p$ components where $r > p$ is evidently proportional to $n_p n_{r-p}$, where the factor of proportionality is independent of the value of the n's. Further the probability of a third body taking part in the encounter is proportional to $\sum_{q=1}^{\infty} n_q$, the factor of proportionality being again independent of the n's. Thus the probability of forming a system with r components in encounters between two systems containing p and $r-p$ components respectively is evidently proportional to $n_p n_{r-p} \sum_{q=1}^{\infty} n_q$. It follows that the number of stars with r components formed between τ and $\tau + \delta\tau$ can be written as:

$$A\delta\tau \left(\sum_{q=1}^{\infty} n_q \right) \left(\sum_{p=1}^{p\,\max} n_p n_{r-p} \right), \tag{43}$$

where $r > 1$, and $p_{\max} = r/2$ if r is even, and $p_{\max} = (r-1)/2$ if r is odd. The factor A in (43) is independent of the value of the n's. Thus we obtain the following general equation for $dn_r/d\tau$ when $r > 1$,

$$dn_r/d\tau = A \left(\sum_{q=1}^{\infty} n_q \right) \left(\sum_{p=1}^{p\,\max} n_p n_{r-p} \right). \tag{44}$$

The equation for $dn_1/d\tau$ is given by differentiating (42).

The solution of the equations (42) and (44) is much simplified by using the observational result that only a small proportion of stars are contained in systems containing more than two components. This means that (42) can be simplified to

$$n_1 + 2n_2 \simeq N, \tag{45}$$

which gives a direct relation between n_1 and n_2. In this case it follows from (44) that

$$\frac{dn_2}{dn_3} = \frac{n_1^2}{n_1 n_2} \simeq \frac{N - 2n_2}{n_2}. \tag{46}$$

An elementary integration of (46) then gives

$$n_3 \simeq -(N/4) \log_e (1 - 2n_2/N) - n_2/2, \tag{47}$$

which determines the relative proportions of binary and triple systems. As an example of the application of (47) we may use the observational result that the total number $2n_2$ of stars contained in binary systems is about equal to the number of single stars. Then it follows immediately from (45) and (47) that

$$\left.\begin{aligned} n_3/n_2 &\simeq \log_e 2 - 0{\cdot}5 = 0{\cdot}1931, \\ n_2 &\simeq N/4. \end{aligned}\right\} \tag{48}$$

Equations (48) show that the ratio of the number of triple systems to the number of binaries should be about one to five, and that the ratio of the number of triple systems to the total number of stars should be about one to twenty. These numerical values are in very good agreement with the observational data given by Kuiper in his survey of the nearest stars.*

* G. P. Kuiper, *loc. cit.*

The value of $n_4(\tau)$ may also be estimated by considerations similar to those given for triple systems. The equation for $dn_4/d\tau$ is

$$dn_4/d\tau = A\left(\sum_{q=1}^{\infty} n_q\right)(n_1 n_3 + n_2^2), \tag{49}$$

which when taken together with the equation for $dn_2/d\tau$ gives

$$\frac{dn_4}{dn_2} = \frac{n_1 n_3 + n_2^2}{n_1^2}. \tag{50}$$

Equation (50) can be solved by writing $n_1 = N - 2n_2$ and using (47) for n_3, thereby giving dn_4/dn_2 as a function of n_2. It is not proposed to discuss equation (50) beyond giving a comparison of the two terms on the right-hand side. According to (48) we have $n_3 \doteq n_2/5$, so that

$$n_1 n_3 \doteq n_1 n_2/5 \doteq 2n_2^2/5,$$

which establishes the result, quoted early in this section, that about 5/7 of quadruple systems are formed in encounters involving two binaries and a third body, the remaining 2/7 of quadruple systems being formed in encounters involving a single star, a triple system and a third body.

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ON THE STRUCTURE OF DISK-SHAPED EXTRAGALACTIC NEBULÆ IV. THE EFFECT OF INTERNEBULAR MATERIAL ON ACCRETION BY STARS

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Summary

The indirect evidence in favour of accretion of interstellar hydrogen by stars is reviewed in 1.1, and the quantitative requirement in the accretion problem are discussed. It is shown in 1.2 that a number of conditions must be fulfilled in order that these requirements be satisfied. Among the necessary conditions for the stars of highest luminosity is that the component parallel to the plane of the nebula of the relative velocity between the star and the interstellar hydrogen must be less than 2 km. per sec. The discussion of this condition forms the main substance of the present paper. The investigation given in Sections 2, 3 and 4 is concerned with the dynamics of an encounter between a nebula (of the type discussed in Papers I and II) and diffuse material lying between the nebulæ. This process is shown to produce changes in the velocities of the stars relative to the interstellar material. These changes lead to conditions in which a particular group of stars satisfy the requirements of the accretion problem. The properties of this group of stars are considered in 4.1 and 4.2.

The results obtained in Section 3 and in subsections 4.1, 4.2 are applied in 4.3 to the problem of the origin of the spiral arms of the nebulæ (it was stated in 4.2 of Paper II that this discussion would be given in the present paper). Finally in Section 5 the question of the stability of axially symmetric nebulæ is considered. The investigation of this latter question, although it bears no direct relation to the work of Sections 2, 3 and 4, is conveniently given at this stage.

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1. *Introduction.*—1.1. The discussions given in Papers I, II and III provide strong indirect evidence in favour of accretion of interstellar material by stars. This evidence may be briefly summarised:—

(*a*) Accretion is necessary to explain the origin of the small group of exceptionally massive and luminous stars that occur in the spiral arms of the nebulæ.

(*b*) The solution of the problem of the luminosity function given in Section 3 of Paper II has the immediate corollary that accretion is necessary to give stars with rotational velocities that are small compared with the value at which rotational instability occurs. Since such stars exist it is natural to infer that accretion has taken place.

(*c*) The evolution of binary systems discussed in Paper III requires accretion to increase the masses of the components of binary systems by a factor that in general is of order 10. The discussion of Paper III also provides evidence that at the time of their condensation the initial masses of the stars are in general less than $M_{\odot}$. This result gives additional confirmation of (*a*) by showing that the highly luminous stars in a nebula have not condensed as such, but have evolved from comparatively small initial masses.

(*d*) The large peculiar velocities possessed by the stars in the nuclei of the nebulæ require that accretion must be negligible for these stars. This provides an explanation of why bright stars of the type mentioned in (*a*) do not occur in the nuclei of the nebulæ.

It is fairly certain by inference from (*a*), (*b*), (*c*), (*d*) that accretion must occur. However, in addition to this inferential argument it is important to verify by deductive argument that the rate of accretion given by direct calculation is sufficiently large to meet the above requirements. The two most important features of these requirements are:

(i) That accretion be able to supply sufficient hydrogen to offset the transmutation of hydrogen into helium occurring within the bright stars. This condition becomes more severe as the mass of a star increases.

(ii) That accretion be sufficiently rapid to give an increase by an appreciable factor in the masses of the components of binary systems. This condition is most severe for binaries of small mass.

The question of satisfying the conditions (i) and (ii) has been considered in detail in former work.* It was found that an extreme example of (i) leads to almost exactly the same conditions as an extreme example of (ii). These conditions are:

(1) That the temperature T_H of the interstellar hydrogen be sufficiently low. This condition is easily satisfied if T_H is sufficiently low for stars to condense.

(2) That ρ, v satisfy

$$\rho/v^3 > \text{a quantity of order } 3.10^{-38}, \qquad (1)$$

where ρ is the density of the interstellar hydrogen in grams per cm.^{3} and v is the velocity in cm. per sec. of the star relative to the interstellar hydrogen. In the previous work an attempt was made to consider the condition (1) by taking a fixed value $v = 4.10^5$ cm. per sec. This leads to the following condition for ρ:

$$\rho > \text{a quantity of order } 2.10^{-21} \text{ gm. per cm.}^3. \qquad (2)$$

The earlier work then proceeded to discuss the question of whether (2) could be regarded as a reasonable condition on ρ. As the information that was then available concerning the value of ρ was somewhat crude it is of interest to review this discussion in the light of the work of the preceding papers.

The density ρ will always be greater than a value of order 10^{-21} gm. per cm.^{3} in the main body of a nebula (where ρt is of order 1/5). Since it is known from Section 4 of Paper I that T_H cannot be appreciably greater than 10,000° abs., this conclusion follows immediately from equation (4) of Paper I. Thus putting $T_H = 10{,}000°$ abs., $\mu = 1$, $\rho t = 1/5$ in this equation gives $\rho = 1{\cdot}1.10^{-21}$ gm. per cm.^{3}, which shows that condition (2) on ρ is satisfied. On the other hand the assumption $v = 4.10^5$ cm. per sec. may not be satisfied in the main body of a nebula. This value of v was based on the observed peculiar velocities of highly luminous stars lying within a distance of about 1000 parsecs from the Sun. Since this region probably lies in the outer fringes of the Galaxy such a value for v cannot necessarily be taken over for the main body of a nebula. For it has been shown in Sections 7 and 8 of Paper I that the thickness of the observed stellar distribution in the main body and in the nucleus of a nebula can only be explained if the stars possess large peculiar velocities in a direction perpendicular to the plane of the nebula. In particular to explain the dimensions of the nuclei of such nebulæ as M31 and M33 it is necessary to suppose that the stars possess peculiar velocities greater than 100 km. per sec. It follows therefore that the condition

* F. Hoyle and R. A. Lyttleton, *Proc. Camb. Phil. Soc.*, **35**, 592, 1939; *Ibid.*, **36**, 325 and 424, 1940; *M.N.*, **101**, 227, 1941.

$v = 4.10^5$ cm. per sec. almost certainly breaks down in the nucleus and may possibly break down in the main body of a nebula. It is an immediate inference that *there can be no highly luminous stars in the nuclei of the nebulæ.*

In the main body of a nebula it is the condition $v = 4.10^5$ cm. per sec. that may break down (ρ being $> 10^{-21}$ gm. per cm.^{3}), whereas in the outer fringes such as the neighbourhood of the Sun the situation is reversed, and it is the condition $\rho > 10^{-21}$ gm. per cm.^{3} that may break down. The reason for this change is that ρt is much less in the outer fringes than in the main body (it is therefore possible to explain the thickness of the stellar distribution in the outer fringes without assuming very large peculiar velocities). For example, if we consider the neighbourhood of the Sun and put $T_H = 100°$ abs. and $\rho t = 10^{-2}$ the value of ρ given by * equation (4) of Paper I is $2{\cdot}8.10^{-22}$ gm. per cm.^{3}. This value is too small to meet the requirement (2) given above by a factor of order 10. It follows therefore that the values $v = 4.10^5$ cm. per sec., $T_H = 100°$ abs., $\rho t = 10^{-2}$ fail to satisfy the inequality (1) by a factor of order 10. This discrepancy can evidently be overcome by taking v to be less than 4.10^5 cm. per sec. The main purpose of the present paper is to discuss the conditions under which this may occur.

1.2. Let the relative velocity v be resolved into a component perpendicular to the plane of the nebula and a component parallel to the plane of the nebula. Consider firstly the component perpendicular to the plane of the nebula. The magnitude of this component has been discussed in 2.3 of Paper III. It was shown that if z is the distance of the star from the plane of the nebula, then the magnitude of the velocity $dz/d\tau$ is given by

$$\sqrt{(8\pi G t\rho/q)}[|z_1| - |z| + (t/q)(e^{-q|z_1|/t} - e^{-q|z|/t})]^{\frac{1}{2}}, \tag{3}$$

where $|z_1|$ is the amplitude of oscillation of the star about the plane of the nebula, and q is a factor of order unity. It can be shown that

$$\frac{1}{|z_1|}\int_0^{|z_1|} \left|\frac{dz}{d\tau}\right| dz,$$

which gives the average value of (3) taken over the period of the oscillation, is of order $\sqrt{(4\pi G\rho)}\,|z_1|$ for any $|z_1| \prec t$. This *average* component of velocity is less than v if

$$|z_1| < v/\sqrt{(4\pi G\rho)}. \tag{4}$$

Now the inequality (1) will be satisfied when $T_H = 100°$ abs., $\rho t = 10^{-2}$, $\rho = 2{\cdot}8.10^{-22}$ gm. per cm.^{3}, provided v is less than about 2.10^5 cm. per sec. Putting $v = 2.10^5$ cm. per sec. and $\rho = 2{\cdot}8.10^{-22}$ gm. per cm.^{3} in (4) gives $|z_1| < 7{\cdot}5.10^{18}$ cm. Thus provided $|z_1|$ satisfies the latter inequality, the inequality (1) will be satisfied so far as the component of relative velocity perpendicular to the plane of the nebula is concerned. It is of interest to note in this connection that the apparent galactic concentration, as judged from the observed stellar distribution, of stars satisfying (4) can be very much greater than the right-hand side of this inequality. For if the hydrogen disk is slightly curved, or possesses irregularities, the stellar distribution may have an apparent thickness that greatly exceeds the amplitude of oscillation of each individual star. Thus it is not

* This value of ρ is large compared with the observational estimates that have been made for the density of the interstellar hydrogen (T. Dunham, *Proc. Amer. Phil. Soc.*, **8**, 277, 1939, and O. Struve and C. T. Elvey, *Ap. J.*, **88**, 364, 1938, and *Ap. J.*, **89**, 119 and 517, 1939). These estimates apply, however, to regions where T_H is of order 10,000° abs. and are accordingly not applicable to regions where T_H is of order 100° abs. For $T_H = 10{,}000°$ abs. and $\rho t = 10^{-2}$, equation (4) of Paper I gives $t = 3{\cdot}7.10^{21}$ cm., $\rho = 2{\cdot}8.10^{-24}$ gm. per cm.^{3}, which is in good agreement with the observational estimates for ρ. Thus there is agreement between theory and observation for the case $T_H = 10{,}000°$ abs., and it is therefore natural to apply the theory with some confidence to the case $T_H = 100°$ abs. where no observational data is available.

necessarily permissible to compare the right-hand side of (4) with the observed thickness of the stellar distribution.

The inequality (4) is not a serious restriction, for even if this inequality is not satisfied there will be no large reduction in the average rate of accretion provided $|z_1| < t$. This can be seen from the following argument. It follows immediately from (3) that there must always be a part of the oscillation in which the value of $|dz/d\tau|$ is less than any given value of v. For since (3) is zero when $|z| = |z_1|$, it is seen that provided $|z_1| - |z|$ is not too large the value of (3) will be less than v however small the value chosen for v. Indeed it is easy to show that in the case where (4) is not satisfied the velocity (3) is less than v for a fraction of order $v/\sqrt{(4\pi G\rho)}.|z_1|$ of the total time. It is necessary for the purpose of accretion that $|z_1|$ be less than a value of order t, since otherwise the star would lie outside the interstellar hydrogen during this fraction of the time. It follows therefore that when

$$t > |z_1| > v/\sqrt{(4\pi G\rho)} \tag{5}$$

the *average* accretion rate is reduced by a factor of about

$$v/\sqrt{(4\pi G\rho)}\,|z_1| \tag{6}$$

below the value that would occur if (4) were satisfied. The smallest value of (6) occurs when $|z_1| = t$. This value is about $\frac{1}{5}$ for the case in which $\rho t = 10^{-2}$, $T_H = 100°$ abs. and $v = 2.10^5$ cm. per sec. Thus even in the extreme case where $|z_1| = t$ the average accretion rate is only reduced by a factor 5 due to (4) not being satisfied.

We may now collect together the conditions necessary for the inequality (1) to be satisfied in the outer fringes of a nebula (for example, in the region of the Galaxy lying within 1000 parsecs from the Sun):

(i) The star is situated in interstellar material in which the temperature T_H is of order 100° abs.
(ii) That $|z_1| \leqslant$ a value of order 10^{19} cm.
(iii) That the component parallel to the plane of the nebula of the relative velocity between the star and the interstellar material be less than about 2.10^5 cm. per sec.

The discussion of (iii) forms the main subject of the present paper. A process is suggested below that leads to a situation in which a small fraction of the total number of stars must satisfy (iii). Those members of this group of stars that also satisfy (i) and (ii) will undergo accretion at a very rapid rate. The result that only a small fraction of stars undergo accretion at a very rapid rate is in accordance with observation. For the very highly luminous stars in a nebula form only about one part in a million of the total number of stars. Indeed it would be a serious disadvantage to the accretion theory if every star were found to undergo accretion at a very rapid rate.

1.3. The following work of the paper is based on the possibility that diffuse gaseous material may exist between the nebulæ. The effect of an encounter between a nebula and such "internebular" material is discussed below. It is found that the momentum of the internebular material striking the nebula will produce relative velocities parallel to the plane of the nebula between the stars and the interstellar material in the nebula. Such relative velocities are of especial interest in the accretion problem discussed in 1.1 and 1.2, for it can be shown that under suitable circumstances the component of relative velocity parallel to the plane of the nebula, existing before the encounter with internebular material, can be cancelled almost completely by the effect of the internebular material. This means that a group of stars obeying certain special properties will satisfy the condition (iii) given at the end of 1.2. It may be noted that although the dynamical investigations given below refer explicitly to the perturbation in the motion of the stars, and of the interstellar material in a nebula, arising from an encounter of

the nebula with internebular material, it is to be expected that any unidirectional force that acts directly on the interstellar material but not on the stars (or *vice versa*) will produce motions similar to those discussed below. Thus the following analysis may be of wider significance than the immediate context in which it is given.

Before proceeding with the discussion of an encounter of a nebula with internebular material, it is of interest to consider the observational evidence in favour of the existence of internebular material. The most important result in this connection is the apparent discrepancy that exists between the two different methods of obtaining the masses of compact nebular clusters. The first method consists in determining the average mass of the nebulæ (such as has been done in detail for M31 and M33) and then simply counting the number of nebulæ within the given cluster. A second method was used by Sinclair Smith * in which the line of sight velocities of 32 nebulæ in the Virgo cluster were found to indicate a velocity of escape from the cluster of about 1500 km. per sec. By using this value, together with the observed dimensions of the cluster, the total mass was obtained and this turned out to be about 100 times greater than the value obtained by the first method. This order of discrepancy, which would appear to lie far outside the range of experimental error, has also been found by Zwicky † in the Coma cluster. The suggestion of Sinclair Smith that the Virgo cluster may contain a large amount of diffuse material lying between the nebulæ seems to be the best interpretation of this apparent discrepancy. The observed dispersions of velocity together with the observed dimensions of the Virgo and Coma clusters suggest a mean density of about 10^{-25} gm. per cm.^{3} for this diffuse material. The above remarks apply explicitly to the compact nebular clusters, but it is clear that if aggregations of diffuse material with density of order 10^{-25} gm. per cm.^{3} are admitted in the dense clusters, then it is reasonable to suppose that aggregations of diffuse material with smaller density will also exist (for example, with density 10^{-28} gm. per cm.^{3}).

2. *The Perturbation Equations for the Interstellar Hydrogen*

2.1.—In this subsection the conditions and assumptions under which the problem is to be solved will be stated.

(1) Let the velocity u of the encounter between the internebular material and the nebula be large compared with the velocity of escape from the nebula. The motion of the internebular material can then to a sufficient approximation be regarded as undisturbed by the gravitational field of the nebula. Accordingly the nebular material will be regarded as striking all points of the nebular disk from the same direction. The numerical value of u will then be taken to be of order 2000 km. per sec.

(2) Let the density σ of the internebular material be uniform, and let σ and u be such that the perturbation in the rotational velocity $V(r)$ of the interstellar material is small compared with the rotational velocity itself.

(3) Let the encounter between the internebular material and the nebula start at time $\tau=0$. In view of (1) it is sufficiently accurate to suppose that the incoming internebular material strikes all points of the nebular disk simultaneously at $\tau=0$.

(4) Let there be no internal motions within the internebular material.

(5) Let the length of the encounter be of the same order as the period of revolution of the stars in the outer fringes of the nebula. For an encounter of this length, and for values of σ satisfying (2), the velocity u of the internebular material relative to the nebula will not change appreciably during the encounter.

(6) Let the nebula be axially symmetric and let the motion of the interstellar material

* S. Smith, *Ap. J.*, **83**, 499, 1936.
† F. Zwicky, *Ap. J.*, **95**, 555, 1942.

be in a steady state at time $\tau = 0$. Then equation (3′) of Paper I will be satisfied at $\tau = 0$.

(7) It can be shown from the theory of atomic collisions that the internebular material striking the nebular disk will be stopped by collisions with the interstellar hydrogen provided $\rho t > 10^{-2}$ and provided the velocity $u < 3000$ km. per sec. For values of u satisfying this inequality and ρt of order 1/5, such as occurs in the main body of M31, M33, or the Galaxy, the incoming internebular material will be stopped in a multistream surface layer. It will be assumed that dynamical currents occur within the interstellar hydrogen that transport the momentum of the incoming internebular material from the multistream region into the main body of the nebula. The component of the velocity of these currents that is important in this connection is the component perpendicular to the plane of the nebula. It will be supposed that through the agency of the currents the momentum of the incoming material is added uniformly throughout the thickness of the disk of interstellar hydrogen. (The existence of these currents can be demonstrated by considering the general hydrodynamical problem, but this forms a major investigation in itself. Since the only result that is here required from such an investigation is the existence of the currents in the interstellar hydrogen, which is in any case a plausible assumption, the general hydrodynamical problem has been regarded as lying outside the scope of the present paper.)

(8) The spherical polar co-ordinates r, θ, ψ will be used in formulating the equations of motion where

(i) The origin of co-ordinates coincides with the centre of gravity of the nebula at all times.

(ii) The plane $\theta = \pi/2$ is the plane of the nebular disk.

(iii) The internebular material approaches the nebula from the direction $\theta = \theta_0$, $\psi = 0$.

2.2. *The Acceleration Acting on the Interstellar Material.*—In this subsection the acceleration acting on the interstellar material, arising from the incoming internebular material, will be determined.

The momentum vector of the incoming material is in the direction $\theta = \theta_0$, $\psi = \pi$ and can be resolved into a component in the direction $\theta = 0$ and a component in the direction $\theta = \pi/2$, $\psi = \pi$. The former component is in a direction perpendicular to the plane of the nebula and consequently will not have any important perturbation on the rotational velocities of either the interstellar material or the stars in the nebula. Thus in a discussion of the first order perturbation of the rotational velocities of the stars and the interstellar material it is not necessary to consider the effect of this component.

The assumption of (7) of 2.1 that the momentum of the incoming internebular material is added uniformly throughout the thickness of the disk of interstellar material, means that for the purposes of the present calculation the disk can be regarded as being of infinitesimal thickness and having a surface mass density ρt. Let A be defined as the area of the nebular disk over which ρt is sufficiently large to stop the incoming internebular material by collisions. The area A will differ somewhat from πr^2_{mb} since r_{mb} was defined in Paper I as the radius of the main body of the nebula (in which ρt is comparable with $M/\pi r^2_{mb}$ where M is the total mass of the nebula). The analysis of the mass distribution * in M33 suggests, however, that A may not be much different from πr^2_{mb}, since for this nebula ρt decreases very rapidly with increasing r at the edge of the disk (it is not possible to say whether this is also the case for M31, since the rotational velocity curve is not available up to the edge of the disk). The total mass of internebular material stopped by the disk is $A\sigma u \cos\theta_0$ per sec. and this material has a momentum

A. B. Wyse and N. U. Mayall, *Ap. J.*, 95, 24, 1942.

$A\sigma u^2 \sin\theta_0 \cos\theta_0$ per sec. in the direction $\theta=\pi/2$, $\psi=\pi$ relative to an unaccelerated observer who is at rest relative to the centre of gravity of the nebula at time $\tau=0$. If $w(\tau)$ is the velocity of the centre of gravity at time τ relative to this observer, then

$$dw(\tau)/d\tau = A\sigma u^2 \sin\theta_0 \cos\theta_0/M, \tag{7}$$

where $w(\tau=0)=0$, and M is the mass of the nebula. Now, since from (5) of 2.1 the velocity u is effectively independent of τ it follows that there is a constant acceleration relative to the observer of magnitude $A\sigma u^2 \sin\theta_0 \cos\theta_0/M$ acting on the nebula in the direction $\theta=\pi/2$, $\psi=\pi$. But the acceleration acting on the interstellar material situated at the point r, ψ is $\sigma u^2 \sin\theta_0 \cos\theta_0/(\rho t)_{r,\psi}$ relative to the observer (the co-ordinates r, ψ are sufficient to define the position of points in the plane of the nebula). Thus since this acceleration is also in the direction $\theta=\pi/2$, $\psi=\pi$, it follows that the acceleration of the interstellar material *relative to the centre of gravity* of the nebula is

$$\sigma u^2 \sin\theta_0 \cos\theta_0 \left(\frac{1}{(\rho t)_{r,\psi}} - \frac{A}{M}\right) \tag{8}$$

in the direction $\theta=\pi/2$, $\psi=\pi$. The expression (8) gives the required acceleration acting on the interstellar material.

2.3. *The Equations of Motion of the Interstellar Hydrogen.*—The numerical value of the acceleration (8) used in the following work is of order 10^{-9} c.g.s. units. This may be compared with the acceleration parallel to the plane of the nebula due to the gas-pressure gradient in the interstellar hydrogen. It was shown in Section 6 of Paper I that the latter acceleration cannot exceed a value of order 10^{-10} c.g.s. units. It follows therefore that the acceleration (8) is taken to be sufficiently large for it to be permissible to omit the gas-pressure terms from the equations determining the motion of the interstellar material. This simplification of the problem suggests that the required equations of motion are:—

$$\begin{aligned}
\frac{d(r^2\dot{\psi})}{d\tau} &= \frac{\sigma r u^2 \sin\theta_0 \cos\theta_0 \sin\psi}{(\rho t)_{r,\psi}}\left(1 - \frac{A(\rho t)_{r,\psi}}{M}\right) + rH(r,\psi,\tau), \\
\frac{d^2r}{d\tau^2} - r\dot{\psi}^2 &= -F(r) - \frac{\sigma u^2 \sin\theta_0 \cos\theta_0 \cos\psi}{(\rho t)_{r,\psi}}\left(1 - \frac{A(\rho t)_{r,\psi}}{M}\right) + G(r,\psi,\tau),
\end{aligned} \tag{9}$$

where it is assumed, for the sake of definiteness in the first of these equations, that the nebula rotates in the direction of increasing ψ, and where $F(r)+G(r,\psi,\tau)$ and $H(r,\psi,\tau)$ are the radial and transverse acceleration at the point r, ψ at time τ due to the gravitational field of the nebula. The accelerations $G(r,\psi,\tau=0)$ and $H(r,\psi,\tau=0)$ are both zero since by (6) of 2.1 the nebula is axially symmetric at $\tau=0$. The physical significance of the accelerations $G(r,\psi,\tau)$ and $H(r,\psi,\tau)$ is that the perturbation of the interstellar hydrogen produces a small change in the distribution of the hydrogen and this in turn produces a small change in the gravitational field of the nebula. It is evident that the ratios $G(r,\psi,\tau)/F(r)$ and $H(r,\psi,\tau)/F(r)$ must be small compared with unity.

The adoption of equations (9) for the motion of the interstellar material involves one important assumption, namely that the solution of (9) must give a velocity vector that has no discontinuities in direction. That is, the streamlines describing the motion of the material must nowhere cross each other. It is possible to prove that this requirement is satisfied in the first approximation but not in higher approximations. To avoid interrupting the main argument this statement will be assumed (the proof although fairly easy is rather long). Thus the equations (9) (which are the equations of a free particle moving in the gravitational field of the nebula under the acceleration (8)) will be adopted for the first order perturbation in the motion of the interstellar hydrogen.

3. *The Velocity of a Star Relative to the Interstellar Hydrogen*

3.1. The complete solution of the problem of the perturbed motion of the interstellar material requires the solution of equations (9) together with Poisson's equation for the gravitational potential. This procedure is unnecessarily complicated, however, for the present purpose of obtaining the velocity of the interstellar material relative to the stars. In particular when we work in terms of relative velocities it is found that the troublesome terms in $G(r, \psi, \tau)$, $H(r, \psi, \tau)$ cancel from the equations (for this to be the case it is only necessary that G/F and H/F be small compared with unity). It is easy to see on physical grounds why this should be so, for the gravitational field of the nebula produces effectively equal accelerations on a star and on the interstellar material lying near the star. Accordingly the gravitational accelerations will not alter the relative velocity between the star and the interstellar material. On the other hand the incoming internebular material will produce changes in the relative velocity between the star and the interstellar material. This is due to the fact that stars being compact dense bodies do not receive any appreciable momentum directly from the incoming internebular material. Thus there is no direct acceleration acting on a star, relative to the observer mentioned above in connection with equation (7), arising from the momentum of the incoming internebular material. It follows therefore that since the internebular material produces an acceleration $A\sigma u^2 \sin\theta_0 \cos\theta_0/M$ of the centre of gravity of the nebula relative to the observer acting in the direction $\theta=\pi/2$, $\psi=\pi$, the star must be accelerated relative to the centre of gravity by amount $A\sigma u^2 \sin\theta_0 \cos\theta_0/M$ in the direction $\theta=\pi/2$, $\psi=0$. Thus it is easy to see that the equations of motion of a star are given by

$$\frac{d(r^2\dot{\psi})}{d\tau} = -\frac{A\sigma u^2 \sin\theta_0 \cos\theta_0 \sin\psi}{M} + rH(r, \psi, \tau),$$

$$\frac{d^2r}{d\tau^2} - r\dot{\psi}^2 = -F(r) + \frac{A\sigma u^2 \sin\theta_0 \cos\theta_0 \cos\psi}{M} + G(r, \psi, \tau), \qquad (10)$$

where r, ψ are co-ordinates that are referred to the centre of gravity of the nebula.

The equations (9) and (10) contain all the information necessary for the solution of the present problem. These equations are awkward to handle in the form in which they have been given, and before continuing the discussion of the relative velocity between the star and the interstellar material it is preferable to reduce (9) and (10) to a form where advantage can be taken of the fact that the σ terms are small compared with $F(r)$. This preliminary reduction of the equations will now be considered.

3.2. *Reduction of the Equations for the Interstellar Hydrogen.*—The first of equations (9) can be solved immediately by successive approximations. Thus consider a small element of the interstellar material that was at r_0, ψ_0 at time $\tau=0$ and let interest be fixed in the subsequent motion of this particular sample of material. The zero order approximation to the solution of the first of equations (9) is

$$(r^2\dot{\psi})_\tau = (r^2\dot{\psi})_{\tau=0} = h(r_0), \qquad (11)$$

where $h(r_0)$ is the angular momentum per unit mass at time $\tau=0$ of the particular material under consideration. Since by (6) of 2.1 this material was in a steady state of motion at time $\tau=0$ it follows that

$$h(r_0) = \sqrt{(r_0^3 F(r_0))} \qquad (12)$$

(that is, the material must have been moving in a circular orbit under the radial gravitational acceleration $F(r_0)$). In the subsequent motion of the material the radial co-ordinate r describing the position of the material must be such that

$$r = r_0.(1 + \text{a quantity of the first order of smallness}).$$

This follows immediately from the fact that the perturbation in the motion of the interstellar material is small. Thus in determining the first order approximation for $(r^2\dot{\psi})_\tau$ it is sufficient to write $(\rho t)_r = (\rho t)_{r_0}$ and $rH(r, \psi, \tau) = r_0 H(r_0, \psi, \tau)$ in the right-hand side of (9). The first order approximation for $(r^2\dot{\psi})_\tau$ is then given by writing

$$\psi = \psi_0 + h(r_0)\tau/r_0^2 \tag{11'}$$

in the right-hand side of (9); this expression for ψ being the zero order approximation obtained from (11). Accordingly $(r^2\dot{\psi})_\tau$ is given to first order by

$$(r^2\dot{\psi})_\tau = h(r_0) + \frac{r_0\sigma u^2 \sin\theta_0 \cos\theta_0}{(\rho t)_{r_0}}\left(1 - \frac{A(\rho t)_{r_0}}{M}\right)\int_0^\tau d\tau' \sin\{\psi_0 + h(r_0)\tau'/r_0^2\}$$
$$+ r_0\int_0^\tau d\tau' H\{r_0, \psi = \psi_0 + h(r_0)\tau'/r_0^2, \tau'\}. \tag{13}$$

Equation (13) gives the angular momentum per unit mass of the particular sample of material under consideration. That is, equation (13) gives the angular momentum per unit mass, to first order, for the material that was at r_0, ψ_0 at time $\tau = 0$. It is more useful in the present problem, however, to give the angular momentum per unit mass of this material when it has moved from the direction ψ_0 to the direction ψ as viewed from the centre of gravity. Thus working in terms of ψ instead of τ it can be shown that (13) is replaced by

$$r^2\dot{\psi} = h(r_0)[1 + \alpha(r_0)(1 - A(\rho t)_{r_0}/M)(\cos\psi_0 - \cos\psi)] + \mathcal{J}(r_0, \psi_0, \psi), \tag{14}$$

where

$$\alpha(r_0) = \sigma u^2 \sin\theta_0 \cos\theta_0 / F(r_0)(\rho t) r_0 \tag{15}$$

and

$$\mathcal{J}(r_0, \psi_0, \psi) = \frac{r_0^3}{h(r_0)}\int_{\psi_0}^{\psi} d\psi' H\{r_0, \psi', \tau = (\psi' - \psi_0) r_0^2/h(r_0)\}, \tag{16}$$

the subscript ψ on $(r^2\dot{\psi})_\psi$ being dropped. It should be noted that the angular variable ψ in (14) must be regarded as increasing continuously. Thus when material has completed a revolution in the nebula and is again in the direction ψ_0 as viewed from the centre of gravity the value of ψ is $\psi_0 + 2\pi$. Similarly when the material has completed n revolutions the value of ψ is $\psi_0 + 2n\pi$.

The next step is to reduce the second of equations (9). This is done by writing $s = 1/r$ and eliminating τ by using (14). After some reduction this gives

$$\frac{d^2 s}{d\psi^2} + s = \frac{F(s)}{h^2(r_0).s^2}\left[1 + \alpha(r_0)\left(1 - \frac{A(\rho t)_{r_0}}{M}\right)(3\cos\psi - 2\cos\psi_0) - \frac{2\mathcal{J}(r_0, \psi_0, \psi)}{h(r_0)}\right]$$
$$+ \frac{G(s, \psi)}{h^2(r_0)s^2} + \text{2nd order terms}. \tag{17}$$

The solution of equation (17) gives the radial co-ordinate r belonging to the material that was at r_0, ψ_0 at $\tau = 0$ when the material has rotated to the angle ψ. Accordingly equation (17) determines the perturbed orbit of this particular sample of material. Now write $r_0 + \zeta(\psi) = 1/s$. Then since the material moves in an approximately circular orbit it follows that

$$\zeta(\psi)/r_0 << 1,$$
$$F(r) = F(r_0) + \zeta F'(r_0) + \text{2nd order terms},$$
$$G(r, \psi) = G(r_0, \psi) + \text{2nd order terms},$$

where $F'(r) = dF(r)/dr$. Again after some reduction equation (17) gives the following differential equation for ζ:—

$$\frac{d^2\zeta}{d\psi^2} + \left[3 + \frac{r_0 F'(r_0)}{F(r_0)}\right]\zeta = r_0\alpha(r_0)\left(1 - \frac{A(\rho t)_{r_0}}{M}\right)(2\cos\psi_0 - 3\cos\psi) + \frac{2r_0\mathcal{J}(r_0, \psi_0, \psi)}{h(r_0)} - \frac{r_0 G(r_0\psi)}{F(r_0)}. \quad (18)$$

The equation (12) is used in this reduction. Equations (14) and (18) give the required reduction of the original equations (9). Equation (18) determines the orbit of the particular sample of material under consideration, and the radial and tangential velocities of the material can be determined from (14) and (18) together with the following equations:—

$$\text{Radial velocity} = \frac{d\zeta}{d\tau} = \frac{d\zeta}{d\psi}\frac{d\psi}{d\tau} \approx \frac{h(r_0)}{r_0^2}\frac{d\zeta}{d\psi}. \quad (19)$$

$$\text{Tangential velocity} = \frac{r^2\dot{\psi}}{r_0 + \zeta(\psi)} \approx \frac{r^2\dot{\psi}}{r_0}\left(1 - \frac{\zeta(\psi)}{r_0}\right). \quad (20)$$

3.3. *Reduction of the Equations for a Star.*—The equations (10) can be reduced to two equations similar to (14) and (18). The procedure is similar to that given in 3.1, except that the star will not in general be in a circular orbit at time $\tau = 0$. Let h' be the angular momentum per unit mass at time $\tau = 0$ of the material of the star about the centre of gravity of the nebula. Then define r_1 by the equation

$$h' = \sqrt{(r_1^3 F(r_1))}. \quad (21)$$

Let the star be at r_0', ψ_0' at time $\tau = 0$, and let the star be moving in an approximately circular orbit at $\tau = 0$ so that

$$r_0' = r_1 \text{ (1 + a small quantity).}$$

The first of equations (10) can be reduced by a procedure similar to that leading to (14). It can be shown in this way that

$$r^2\dot{\psi} = h'[1 - A\alpha(r_0')(\rho t)_{r_0}(\cos\psi_0 - \cos\psi)/M] + \mathcal{J}'(r_0', \psi_0', \psi) + \text{2nd order terms}, \quad (22)$$

where

$$\mathcal{J}'(r_0', \psi_0', \psi) = \frac{r_0'^3}{h'}\int_{\psi_0'}^{\psi} d\psi' H\{r_0', \psi', \tau = (\psi' - \psi_0')r_0'^2/h'\}. \quad (23)$$

Furthermore, the second of the equations (10) can be reduced to an equation similar to (18). Thus if $r = r_1 + \xi(\psi)$ it can be shown that

$$\frac{d^2\xi}{d\psi^2} + \left[3 + \frac{r_1 F'(r_1)}{F(r_1)}\right]\xi = -Ar_1\alpha(r_0)(\rho t)_{r_0}(2\cos\psi_0' - 3\cos\psi)/M - \frac{r_1 G(r_1\psi)}{F(r_1)} + \frac{2r_1\mathcal{J}'(r_1, \psi_0', \psi)}{h'}, \quad (24)$$

where equation (21) is used in making this reduction. Equations (22), (24) give the required equations for the perturbed motion of the star.

3.4. *The Relative Velocity between a Star and the Interstellar Hydrogen.*—Let a star satisfying equations (22) and (24) be coincident at the point r, ψ with the interstellar material that satisfies (14) and (18). Then it follows that

$$r_0 + \zeta(\psi) = r_1 + \xi(\psi). \quad (25)$$

It should be noted that this equation will only be satisfied for the particular value of ψ

at which the star is coincident with the sample of interstellar material that was at r_0, ψ_0 at time $\tau = 0$. Equation (25) shows that

$$r_1 = r_0(1 + \text{a quantity of the first order of smallness}).$$

Moreover, $1 - r_0'/r_0$ and $1 - \psi_0'/\psi_0$ are also quantities of first order. It follows therefore that the various quantities on the right-hand side of (24) must satisfy the following conditions:—

$$\begin{aligned} r_1\alpha(r_0')(\rho t)_{r_0'}(2\cos\psi_0' - 3\cos\psi) &= r_0\alpha(r_0)(\rho t)_{r_0}(2\cos\psi_0 - 3\cos\psi) + \text{2nd order terms},\\ r_1 G(r_1, \psi)/F(r_1) &= r_0 G(r_0, \psi)/F(r_0) + \text{2nd order terms},\\ 2r_1\mathcal{J}'(r_1, \psi_0', \psi)/h' &= 2r_0\mathcal{J}(r_0, \psi_0, \psi)/h(r_0) + \text{2nd order terms},\\ r_1 F'(r_1)\xi/F(r_1) &= r_0 F'(r_0)\xi/F(r_0) + \text{2nd order terms}, \end{aligned} \tag{26}$$

since α, G, $\mathcal{J}$ and ξ are themselves quantities of the first order of smallness. It is seen therefore that if second order terms are neglected equation (24) can be written as

$$\frac{d^2\xi}{d\psi^2} + \left[3 + \frac{r_0F'(r_0)}{F(r_0)}\right]\xi = -Ar_0\alpha(r_0)(\rho t)_{r_0}(2\cos\psi_0 - 3\cos\psi)/M - \frac{r_0 G(r_0, \psi)}{F(r_0)} + \frac{2r_0\mathcal{J}(r_0, \psi_0, \psi)}{h(r_0)}. \tag{27}$$

Subtracting (27) from (18) gives

$$\frac{d^2(\zeta - \xi)}{d\psi^2} + \left[3 + \frac{r_0F'(r_0)}{F(r_0)}\right](\zeta - \xi) = r_0\alpha(r_0)(2\cos\psi_0 - 3\cos\psi), \tag{28}$$

the terms in A/M, G and $\mathcal{J}$ cancelling. This equation can be solved immediately to give

$$\frac{\zeta - \xi}{r_0} = e\cos\left[\sqrt{\{3 + r_0F'(r_0)/F(r_0)\}}\psi + \delta\right] + \alpha(r_0)\left[\frac{2\cos\psi_0}{3 + r_0F'(r_0)/F(r_0)} - \frac{3\cos\psi}{2 + r_0F'(r_0)/F(r)}\right], \tag{29}$$

where e, δ are constants of integration. The radial component $v_r(r, \psi)$ of the velocity of the interstellar material relative to the star, when the star is at the point r, ψ, is given by

$$\frac{d(\zeta - \xi)}{d\tau} = \frac{h(r_0)}{r_0^2}\frac{d(\zeta - \xi)}{d\psi}. \tag{30}$$

Now since $r = r_0(1 + \text{first order quantity})$ in (30) it follows that r_0 can be replaced by r in both (29) and (30). That is

$$\frac{\zeta - \xi}{r} = e\cos\left[\sqrt{\{3 + rF'(r)/F(r)\}}\psi + \delta\right] + \alpha(r)\left[\frac{2\cos\psi_0}{3 + rF'(r)/F(r)} - \frac{3\cos\psi}{2 + rF'(r)/F(r)}\right] \tag{29'}$$

and

$$v_r(r, \psi) = (h(r)/r^2).d(\zeta - \xi)/d\psi. \tag{30'}$$

The value of $d(\zeta - \xi)/d\psi$ is given immediately by (29') so that $v_r(r, \psi)$ becomes

$$v_r(r, \psi) = \sqrt{\{rF(r)\}}\{-e\sqrt{\{3 + rF'/F\}}\sin\left[\sqrt{\{3 + rF'/F\}}\psi + \delta\right] + 3\alpha(r)\sin\psi/(2 + rF'/F)\}. \tag{31}$$

Equation (31) gives an expression for the radial component of velocity of a star relative to the interstellar material at the time when the star is at the point r, ψ. The significance of the constants e, δ will be discussed below in Section 4.

It is also easy to obtain the tangential component $v_\psi(r, \psi)$ of the relative velocity between the interstellar material and the star (the tangential velocity being the velocity

in the direction of increasing ψ). The tangential component of the velocity of the interstellar material is given by (14) and (18) and is

$$[h(r_0)\{1+\alpha(r_0)(1-A(\rho t)_{r_0}/M)(\cos\psi_0-\cos\psi)\}+\mathcal{J}(r_0\psi_0\psi)]/[r_0+\zeta(\psi)], \tag{32}$$

whilst the tangential velocity of the star is

$$[h'\{1-A\alpha(r_0')(\rho t)_{r_0}/M.(\cos\psi_0-\cos\psi)\}+\mathcal{J}'(r_0',\psi_0',\psi)]/[r_1+\xi(\psi)]. \tag{33}$$

In the latter expression we write

$$\alpha(r_0')(\rho t)_{r_0}(\cos\psi_0'-\cos\psi)=\alpha(r_0)(\rho t)_{r_0}(\cos\psi_0-\cos\psi)+\text{2nd order terms},$$
$$\mathcal{J}'(r_0'\psi_0'\psi)/[r_1+\xi(\psi)]=\mathcal{J}(r_0\psi_0\psi)/[r_0+\zeta(\psi)]+\text{2nd order terms}, \tag{34}$$

so that the difference between (32) and (33) is

$$v_\psi(r,\psi)=h(r_0)[1+\alpha(r_0)(1-A(\rho t)_{r_0}/M)(\cos\psi_0-\cos\psi)]/[r_0+\zeta(\psi)]$$
$$-h'[1-A(\rho t)_{r_0}\alpha(r_0)(\cos\psi_0-\cos\psi)/M]/[r_1+\xi(\psi)]. \tag{35}$$

By using (12) and (21) for $h(r_0)$ and h' respectively and writing

$$r_1=r_0+\zeta(\psi)-\xi(\psi),$$
$$F(r_1)=F(r_0)[1+(\zeta-\xi)F'(r_0)/F(r_0)]+\text{2nd order terms},$$
$$r=r_1(1+\text{quantity of 1st order}),$$
$$r=r_0(1+\text{quantity of 1st order}),$$

we finally obtain

$$v_r(r,\psi)=\sqrt{rF(r)}\{-e/2.(3+rF'(r)/F(r))\cos[\sqrt{(3+rF'/F)}\psi+\delta]$$
$$+\alpha(r)(5+rF'/F)\cos\psi/2(2+rF'/F)\}. \tag{36}$$

Equations (31) and (36) give the solution to the present problem. Thus if v_r, v_ψ are known for any particular value of ψ, then e, δ can be determined from a pair of simultaneous equations. Then knowing the values of e, δ, the values of v_r, v_ψ can be obtained for any other value of ψ (the functions F, F' being known for a given nebula).

4. *Application to Accretion of Interstellar Hydrogen*

4.1. The process discussed in Sections 2 and 3 may have an important effect on accretion by stars if v_r, v_ψ are simultaneously zero for some value of ψ, ψ_1 say. Then from (31) and (36) it is seen that ψ_1 must satisfy the following equations:—

$$3\sin\psi_1/(2+rF'/F)-e/\alpha\sqrt{\{3+rF'/F\}}\sin[\sqrt{\{3+rF'/F\}}\psi_1+\delta]=0,$$
$$\cos\psi_1(5+rF'/F)/(2+rF'/F)-e/\alpha(3+rF'/F)\cos[\sqrt{\{3+rF'/F\}}\psi_1+\delta]=0. \tag{37}$$

The solution of the equations (37) is very much simplified if the slight approximation is made of writing $3\sqrt{\{3+rF'/F\}}/(5+rF'/F)$ equal to unity. This expression is accurately unity for $rF'/F=1$, is 1·04 for $rF'/F=0$, and is 1·06 for $rF'/F=-1$. Then it follows immediately by eliminating e from the equations (37) that

$$\psi_1\sqrt{\{3+rF'/F\}}+\delta\approx 2n\pi, \tag{38}$$

where n is an integer. Moreover, to the same degree of approximation

$$\frac{e}{\alpha}\approx\frac{5+rF'/F}{(2+rF'/F)(3+rF'/F)}\approx\frac{3}{(2+rF'/F)\sqrt{(3+rF'/F)}}.$$

The values of e, δ given by these two equations can be substituted in (31) and (36) to give

$$v_r(r, \psi)(2+rF'/F)/3\alpha(r)\sqrt{(rF)} = \sin\psi - \sin[\sqrt{(3+rF'/F)}\psi - (\sqrt{(3+rF'/F)}-1)\psi_1],$$
$$2v_\psi(r, \psi)(2+rF'/F)/\alpha(r)\sqrt{(rF)}(5+rF'/F)$$
$$= \cos\psi - \cos[\sqrt{(3+rF'/F)}\psi - (\sqrt{(3+rF'/F)}-1)\psi_1], \quad (39)$$

which determine v_r, v_ψ in terms of ψ_1 for all values of ψ. Now $v_r(\tau=0)$, $v_\psi(\tau=0)$, which are the radial and tangential components of relative velocity between the interstellar material and the star at time $\tau=0$, should be regarded as given quantities. Thus (39) gives the following equations which are to be regarded as determining whether for given ψ_0, $v_r(r, \psi_0, \tau=0)$, $v_\psi(r, \psi_0, \tau=0)$, a solution for ψ_1 exists:

$$v_r(r, \psi_0, \tau=0)(2+rF'/F)/3\alpha(r)\sqrt{(rF)} = \sin\psi_0 - \sin\{\sqrt{(3+rF'/F)}\psi_0 - (\sqrt{(3+rF'/F)}-1)\psi_1\},$$
$$2v_\psi(r, \psi_0, \tau=0)(2+rF'/F)/\alpha(r)\sqrt{(rF)}(5+rF'/F)$$
$$= \cos\psi_0 - \cos\{\sqrt{(3+rF'/F)}\psi_0 - (\sqrt{(3+rF'/F)}-1)\psi_1\}. \quad (40)$$

If a solution of (40) exists, then a star possessing the given relative velocities $v_r(r, \psi_0, \tau=0)$, $v_\psi(r, \psi_0, \tau=0)$ and co-ordinates r, ψ_0 at time $\tau=0$ will have zero values of both v_r, v_ψ when it has reached angle ψ_1 (as viewed from the centre of gravity of the nebula). Squaring and adding the two equations (37) gives

$$(2+rF'/F)^2/\alpha^2 rF\{v_r^2(r, \psi_0, \tau=0)/9 + 4v_\psi^2(r, \psi_0, \tau=0)/(5+rF'/F)^2\}$$
$$= 2\{1 - \cos[(\sqrt{(3+rF'/F)}-1)(\psi_0-\psi_1)]\}, \quad (41)$$

and this equation has a series of solutions for ψ_1 provided

$$v_r^2(r, \psi_0, \tau=0)/9 + 4v_\psi^2(r, \psi_0, \tau=0)/(5+rF'/F)^2 \leqslant 4\alpha^2 rF/(2+rF'/F)^2. \quad (42)$$

The inequality (42) is the condition that $v_r(r, \psi_0, \tau=0)$, $v_\psi(r, \psi_0, \tau=0)$ must satisfy in order that there may be some value of ψ, ψ_1 say, at which both v_r, v_ψ are zero. It is easy to show that if $v_r(r, \psi_0, \tau=0)$, $v_\psi(r, \psi_0, \tau=0)$ satisfy (42) and ψ_1 is the smallest value of ψ satisfying (40), then there will be a series of values of ψ given by

$$\psi = \psi_1 + 2m\pi/(\sqrt{(3+rF'/F)}-1), \quad (43)$$

where $m=0, 1, 2, 3, \ldots$, etc., all of which satisfy (40). The values of v_r, v_ψ will both be zero at all the values of ψ satisfying (43), so that there will be a series of values of ψ that are of interest in the accretion problem.

4.2. We can now relate the work given in 4.1 with the discussion of 1.1 and 1.2. The total component of relative velocity parallel to the plane of the nebula between a star and the interstellar material is $\sqrt{(v_r^2+v_\psi^2)}$. Accordingly it is necessary for v_r, v_ψ to be simultaneously small in order to satisfy the requirement (iii) given at the end of 1.2.

The encounter between the nebula and the internebular material is not regarded as lasting for a sufficient length of time (see (5) of 2.1) for accretion of interstellar hydrogen by stars to be important during the encounter itself. It is important, however, to consider stars that have $v_r=v_\psi=0$ at the end of the encounter, for after the encounter there will be no accelerations of first order capable of altering the values of v_r, v_ψ (close collisions between the stars will in general take a long time to operate). This statement can be proved formally. Thus the equations for v_r, v_ψ after the encounter are evidently of the form given by putting $\alpha=0$ in (31) and (36). The constants e, δ must of course be given different values after the encounter to their values during the encounter. Accordingly the velocities v_r, v_ψ after the encounter can be written in the form

$$v_r(r, \psi)/\sqrt{(rF)} = -\bar{e}\sqrt{(3+rF'/F)}\sin[\sqrt{(3+rF'/F)}\psi + \bar{\delta}]$$
$$v_\psi(r, \psi)/\sqrt{(rF)} = -\bar{e}/2.(3+rF'/F)\cos[\sqrt{(3+rF'/F)}\psi + \bar{\delta}], \quad (44)$$

where the constants $\bar{e}$, $\bar{\delta}$ are to be determined from the values of v_r, v_ψ, ψ that occur at the end of the encounter. It follows immediately from (44) that if

$$v_r = v_\psi = 0 \tag{45}$$

at the end of the encounter, then $\bar{e}$ must be zero. Thus in this case v_r and v_ψ must be zero at all times after the encounter (so far as the first order approximation is concerned). It follows therefore that stars satisfying (45) at the end of the encounter will have $\sqrt{(v_r^2 + v_\psi^2)} \approx 0$ for a very considerable period of time after the end of the encounter (that is, for a period of time that is large compared with the time of revolution of the star in the nebula). Accordingly it is important to consider the conditions under which a star satisfies (45) at the end of the encounter. This question will now be discussed.

We consider the conditions that the velocities $v_r(r, \psi_0, \tau = 0)$, $v_\psi(r, \psi_0, \tau = 0)$ must satisfy at the beginning of the encounter in order that $v_r = v_\psi = 0$ at the end of the encounter. Let $\bar{\tau}$ be the length of the encounter. Then to a sufficient approximation the direction of the star at the end of the encounter (as viewed from the centre of gravity of the nebula) is given by (11′) and is

$$\psi_0 + \bar{\tau} h(r)/r^2 = \psi_0 + \bar{\tau}\sqrt{(F/r)}. \tag{46}$$

The discussion of 4.1 shows that both v_r, v_ψ will be zero at this time provided (46) is one of the series of values given by (43). That is, if

$$\psi_0 + \bar{\tau}\sqrt{(F/r)} = \psi_1 + 2m\pi/(\sqrt{(3 + rF'/F)} - 1), \tag{47}$$

where m is any of the values 0, 1, 2, . . . Equation (47) can be regarded as determining ψ_1 in terms of ψ_0 and $\bar{\tau}$. Substituting for ψ_1 from (47) in (40) gives the following conditions for $v_r(r, \psi_0, \tau = 0)$ and $v_\psi(r, \psi_0, \tau = 0)$:—

$$\frac{(2 + rF'/F)v_r(r, \psi_0, \tau = 0)}{3\alpha\sqrt{(rF)}} = \sin\psi_0 - \sin\{\psi_0 - \bar{\tau}\sqrt{(F/r)}(\sqrt{(3 + rF'/F)} - 1)\},$$

$$\frac{2(2 + rF'/F)v_\psi(r, \psi_0, \tau = 0)}{\alpha\sqrt{(rF)}(5 + rF'/F)} = \cos\psi_0 - \cos\{\psi_0 - \bar{\tau}\sqrt{(F/r)}(\sqrt{(3 + rF'/F)} - 1)\}. \tag{48}$$

Squaring and adding these equations gives

$$\{(2 + rF'/F)^2/\alpha^2 rF\} \cdot \{v_r^2(r, \psi_0, \tau = 0)/9 + 4v_\psi^2(r, \psi_0, \tau = 0)/(5 + rF'/F)^2\}$$
$$= 2[1 - \cos\{\bar{\tau}\sqrt{(F/r)}(\sqrt{(3 + rF'/F)} - 1)\}], \tag{49}$$

which provides the required condition on $v_r(r, \psi_0, \tau = 0)$, $v_\psi(r, \psi_0, \tau = 0)$.

Equation (49) has a number of interesting consequences. In general $\bar{\tau}$ will be such that the right-hand side of (49) is of order unity. Thus in general we must have

$$\{v_r^2(r, \psi_0, \tau = 0)/9 + 4v_\psi^2(r, \psi_0, \tau = 0)/(5 + rF'/F)^2\}^{\frac{1}{2}} \approx \alpha\sqrt{(rF)}/(2 + rF'/F). \tag{50}$$

In the main body of a nebula $rF'/F \approx 1$ and in this case (50) becomes

$$[v_r^2(r, \psi_0, \tau = 0) + v_\psi^2(r, \psi_0, \tau = 0)]^{\frac{1}{2}} \approx \alpha\sqrt{(rF)}. \tag{50′}$$

Now from equation (3′) of Paper I the quantity $\sqrt{(rF)}$ is the rotational velocity $V(r)$ of the interstellar material in the nebula (before the encounter $V(r)$ is strictly equal to $\sqrt{(rF)}$, while during and after the encounter $V(r)$ is equal to $\sqrt{(rF)}$ in the zero order approximation). It follows therefore that all stars undergoing large accretion *after the encounter* must have had peculiar velocities parallel to the plane of the nebula of order $\alpha V(r)$ *before the encounter*. So far as order of magnitude is concerned this latter statement applies also to other values of rF'/F. It follows immediately that if α is sufficiently large we shall have a situation in which

(*a*) All stars undergoing large accretion after the end of the encounter between the nebula and the internebular material are stars that had a small rate of accretion before the encounter.

(*b*) All stars with a large accretion rate before the encounter (that is $\sqrt{(v_r^2+v_\psi^2)}$ close to zero) will have a small accretion rate after the encounter.

For the region of the Galaxy lying within 1000 parsecs of the Sun we may put $V(r) \approx 300$ km. per sec. Then the situation described in (*a*) and (*b*) will arise in this region for any value of α appreciably greater than 10^{-2} (the condition (iii) of 1.2 requires $\sqrt{(v_r^2+v_\psi^2)}$ to be $< 2.10^5$ cm. per sec. in order to give a sufficiently large accretion rate for the highly luminous stars).

It follows from the above discussion that provided

(i) The special case

$$1-\cos\left[\bar{\tau}\sqrt{(F/r)}(\sqrt{(3+rF'/F)}-1)\right] << 1$$

is excluded,

(ii) That $\alpha(r)$ is sufficiently large (a value of 3.10^{-2} would suffice in nearly all cases),

then the effect of an encounter of the nebula with internebular material is to alter completely the identity of the stars that undergo accretion at a very rapid rate. Thus (*a*) means that a new group of stars (probably of initial mass $< M_\odot$) will increase rapidly in mass and will eventually form a new group of highly luminous stars. Binary systems will evolve rapidly in this group of stars. On the other hand after the encounter the stars that accreted rapidly before the encounter will, if highly luminous, rapidly exhaust their supply of hydrogen. Such stars will then enter a period of gravitational contraction which will continue until

(1) If the mass of the star is less than Chandrasekhar's limit * a white dwarf equilibrium state may be reached.

(2) If the mass of the star is greater than Chandrasekhar's limit the contraction will continue until a state of rotational instability is reached, when there will occur † either a series of nova outbursts or a supernova outburst similar to that discussed by Baade and Minkowski ‡ in connection with the Crab nebula.

4.3. *The Spiral Arms of the Nebulæ.*—It was shown in 4.2 of Paper II that an adequate theory of the development of the spiral arms of the nebula can be formulated if any one of three following alternative requirements is satisfied:—

(1) That massive stars condense at a given time $\tau=0$ say along the diameter of the nebula given by the angles $\psi=\psi'$ and $\psi=\psi'+\pi$ and that no other massive stars condense in the nebula.

(2) That the only stars becoming massive through accretion of interstellar material lie along the diameter given by ψ' and $\psi'=\pi$ at time $\tau=0$.

(3) That although massive stars may occur throughout the nebula, only those stars that lie along the diameter given by ψ' and $\psi'+\pi$ at time $\tau=0$ will undergo accretion of interstellar hydrogen at a sufficient rate to avoid exhausting their supply of hydrogen.

A possibility of satisfying the requirement (3) arises out of the work of 4.1 and 4.2. It was shown in 4.2 that the only stars that undergo accretion at a rapid rate after an encounter of the nebula with internebular material must have values of $v_r(r, \psi_0, \tau=0)$, $v_\psi(r, \psi_0, \tau=0)$ that satisfy equations (48). These equations will now be considered in greater detail.

* S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, p. 437.

† The cause of the difference between a nova outburst and a supernova outburst will be discussed in a future paper.

‡ W. Baade, *Ap. J.*, **96**, 188, 1942; R. Minkowski, *Ap. J.*, **96**, 199, 1942.

The velocities v_r, v_ψ before the encounter must satisfy equations similar to (44). That is

$$v_r(r, \psi)/\sqrt{(rF)} = -e'\sqrt{(3+rF'/F)}\sin\{\sqrt{(3+rF'/F)}\psi+\delta'\},$$
$$v_\psi(r, \psi)/\sqrt{(rF)} = -(e'/2).(3+rF'/F)\cos\{\sqrt{(3+rF'/F)}\psi+\delta'\}, \quad (51)$$

where the constants e', δ' are in general different both from $\bar{e}$, $\bar{\delta}$ in (44) and e, δ in (31) and (36). From (51) it follows that $v_r(r, \psi_0, \tau=0)$, $v_\psi(r, \psi_0, \tau=0)$ can be written as

$$v_r(r, \psi_0, \tau=0)/\sqrt{rF} = -e'\sqrt{3+rF'/F}\sin\{\sqrt{(3+rF'/F)}\psi_0+\delta'\},$$
$$v_\psi(r, \psi_0, \tau=0)/\sqrt{(rF)} = -e'/2(3+rF'/F)\cos\{\sqrt{(3+rF'/F)}\psi_0+\delta'\}. \quad (52)$$

Writing these expressions for $v_r(r, \psi_0, \tau=0)$, $v_\psi(r, \psi_0, \tau=0)$ in (48) gives

$$-(e'/3a)\sqrt{(3+rF'/F)}.(2+rF'/F)\sin\{\sqrt{(3+rF'/F)}\psi_0+\delta'\}$$
$$=2\sin\{(\bar{\tau}/2)\sqrt{(F/r)}(\sqrt{(3+rF'/F)}-1)\}\cos\{\psi_0-\bar{\tau}/2\sqrt{(F/r)}(\sqrt{(3+rF'/F)}-1)\}, \quad (53)$$

together with

$$\frac{e'(3+rF'/F)(2+rF'/F)}{a(5+rF'/F)}\cos\{\sqrt{(3+rF'/F)}\psi_0+\delta'\}$$
$$=2\sin\{\bar{\tau}/2\sqrt{(F/r)}(\sqrt{(3+rF'/F)}-1)\}\sin\{\psi_0-\bar{\tau}/2\sqrt{(F/r)}(\sqrt{(3+rF'/F)}-1)\}. \quad (54)$$

In the main body of a nebula we may put $rF'/F \approx 1$ and the equations (53) and (54) reduce to

$$(e'/a).\sin(2\psi_0+\delta') = -\sin\{(\bar{\tau}/2)\sqrt{(F/r)}\}\cos\{\psi_0-(\bar{\tau}/2)\sqrt{(F/r)}\},$$
$$e'/a\cos(2\psi_0+\delta') = \sin\{(\bar{\tau}/2)\sqrt{(F/r)}\}\sin\{\psi_0-(\bar{\tau}/2)\sqrt{(F/r)}\}. \quad (55)$$

Squaring and adding these equations gives

$$e'^2 = a^2\sin^2\{(\bar{\tau}/2)\sqrt{(F/r)}\}, \quad (56)$$

which is a condition equivalent to (49). In addition to the condition (56) on e' a condition on ψ_0 can be obtained by dividing the two equations (55), which gives

$$\cos\{\psi_0+\delta'+(\bar{\tau}/2)\sqrt{(F/r)}\} = 0. \quad (57)$$

Equation (57) has a series of solutions for ψ_0 given by

$$\psi_0 = (2n+1)\pi/2-(\delta'+(\bar{\tau}/2)\sqrt{(F/r)}), \quad (58)$$

where n is any positive or negative integer. Now (58) has only two solutions for ψ_0 lying between 0 and 2π. These solutions differ by π and may be written as ψ' and $\psi'+\pi$, where the value of ψ' depends on δ' and $\bar{\tau}$. It follows therefore that if all stars satisfying (56) have the same δ', then all stars that undergo rapid accretion after the encounter must lie along one of the two directions $\psi_0=\psi'$ or $\psi_0=\psi'+\pi$ at time $\tau=0$. Thus we have a situation in which the requirement (3) of 4.2 is satisfied.

This discussion does not of course provide a complete solution of the question, since it is still necessary to explain why all stars satisfying (56) should have the same value of δ'. Such a situation could only arise if the deviation of the orbits of the stars from the circular form arose from a common cause, such as, for example, from a suitable deviation from axial symmetry in the distribution of the interstellar material in the nebula. The present discussion has been given, not as a solution of the problem, but in order to show how the situation discussed in 4.2 of Paper II may possibly arise.

The above remarks refer explicitly to the case $rF'/F=1$. A similar investigation can be given for other values of rF'/F. It can be shown that the two important values

of ψ_0 differ by $\pi/(\sqrt{(3+rF'/F)}-1)$ instead of by π as in the case of $rF'/F=1$. This means that the spiral arms, developing according to the theory of 4.2 of Paper II, will only be diametrically opposite when $rF'/F=1$. However, since rF'/F can usually be regarded as lying between 0·5 and 1 in the main body of a nebula the deviation of the arms from exact symmetry will not normally be very marked.

4.4. The conclusions of 4.2 and 4.3 require $\alpha(r)$ to be greater than 10^{-2} (this was shown in 4.2. This condition on $\alpha(r)$ gives a condition on the density σ of the internebular material. Thus it follows from (15) that if $\alpha(r)$ is to be $\geqslant 10^{-2}$, then σ must satisfy the inequality

$$\sigma \geqslant 10^{-2}F(r)(\rho t)_r/u^2 \sin\theta_0 \cos\theta_0. \tag{59}$$

Now in a typical nebula the radial gravitational acceleration is of order 10^{-7} c.g.s. units. It follows therefore from (59) that if $3\cdot10^{16}$ is taken as a reasonable value for $u^2 \sin\theta_0 \cos\theta_0$ then σ must satisfy

$$\sigma \geqslant 3{\cdot}33\cdot10^{-26}(\rho t)_r. \tag{59'}$$

The inequality (59′) leads immediately to the following conclusions:—

(*a*) The density σ of the internebular material must be of order $3{\cdot}33\cdot10^{-27}$ gm. per cm.^{3} for an encounter between the nebula and the internebular material to have an important effect on the relative velocity between the stars and the interstellar material *in the main body of the nebula*, where $(\rho t)_r$ is of order 10^{-1}.

(*b*) The density σ must be of order $3{\cdot}33\cdot10^{-28}$ gm. per cm.^{3} for an encounter to be important for the stars in the *outer fringes of the nebula*, where $(\rho t)_r$ is of order 10^{-2}.

Thus since the average density of the internebular material is believed for cosmological reasons to be less than 10^{-28} gm. per cm.^{3}, this result suggests that encounters of importance for the stars in the outer fringes of a nebula will be more frequent than encounters of importance for the stars in the main body of the nebula.

The thermal energy added to the interstellar material by the incoming internebular material is $\sigma u^3 \cos\theta_0/2$ ergs per sec. per cm.^{2} of the nebular disk. A numerical estimate for this energy can be obtained if σ and u are known. As an example write $u=2\cdot10^8$ cm. per sec., $\cos\theta_0=1/2$, and consider the value $\sigma=3{\cdot}33\cdot10^{-28}$ gm. per cm.^{3} mentioned in conclusion (*b*) of the previous paragraph. This gives an energy addition to the interstellar material of $6{\cdot}66\cdot10^{-4}$ ergs per sec. per cm.^{2} of the nebular disk. This value is closely comparable with the *minimum* rate of addition of thermal energy that was shown in Section 5 of Paper I to be necessary to maintain the temperature T_H of the interstellar hydrogen at a value of order 10,000° abs. It is therefore possible that the addition of energy by the internebular material may raise T_H to a value of order 10,000° abs. in regions where the density of the interstellar dust is not too high. It seems possible that the regions of hydrogen emission in the Galaxy observed by Struve and Elvey * and the emission lines described by Mayall for the extragalactic nebulæ † may be partly due to the present process rather than to excitation by stars of very high surface temperature.

5. *Regular and Irregular Nebulæ.*—The direct observation of regular nebulæ shows that in the main body the period of a star increases with increasing central distance. This requires rF'/F to be less than unity in the main body of a regular nebula (although rF'/F need not be much less than unity, as is shown by the observations for M31 and M33). It is of interest to consider why this should be the case.

Let the distribution of the interstellar material (which forms the main contribution to the mass of the nebula) be approximately axially symmetric. Consider a particular element of interstellar material and let h be its angular momentum per unit mass. A

* O. Struve and C. T. Elvey, *Ap. J.*, **88**, 364, 1938; and **89**, 119 and 517, 1939.

† N. U. Mayall, *Lick. Obs. Bull.*, No. 497, 1939.

slight deviation from axial symmetry in the distribution of material within the nebula will in general result in the gravitational field having a tangential component which will produce a slow change of h with time. The first step in the following investigation is to find the dependence of the orbital period on the value of h. The solution of this question is much simplified if h does not change appreciably during the period of the material, for in this case it is permissible to regard h as being constant throughout any given period. This simplification is made in the discussion given below.

Since gas-pressure terms can be neglected, the equation of the orbit of the material has the same form as the equation of a free particle moving in an almost circular orbit under the central force $F(r)$. This equation is well known in elementary dynamics and is

$$r=\bar{r}+f\cos\left[\sqrt{\{3+\bar{r}F'(r)/F(\bar{r})\}}\psi+\lambda\right], \tag{60}$$

where f, λ are independent of ψ and $\bar{r}$ is defined by

$$h=\sqrt{\{\bar{r}^3F(\bar{r})\}}. \tag{61}$$

The tangential velocity of the material is h/r and its angular velocity $d\psi/d\tau$ is h/r^2. Thus the period $P(h)$ of the material in the nebula is

$$\frac{1}{h}\int_0^{2\pi}[\bar{r}+f\cos\{\sqrt{(3+\bar{r}F'/F)}\psi+\lambda\}]^2d\psi\approx\frac{2\pi\bar{r}^2}{h}, \tag{62}$$

neglecting the small terms in f. Now since h must change slowly with time in an approximately axially symmetric nebula the period P will also change slowly with time. Let h change to $h+\Delta h$ in a given time interval that is long compared with P. Then P will change to $P+\Delta P$ where the increment ΔP is given by the equations

$$\Delta P=(4\pi\bar{r}/h)(\Delta\bar{r}-\bar{r}\Delta h/2h),$$
$$h+\Delta h=\sqrt{\{(\bar{r}+\Delta\bar{r})^3F(\bar{r}+\Delta\bar{r})\}} \tag{63}$$

(any slight change that occurs in the function $F(r)$ during the given time interval being neglected). From equation (63) we have

$$\Delta P/\Delta h=2\pi(\bar{r}/h)^2(1-\bar{r}F'/F)/(3+\bar{r}F'/F), \tag{64}$$

from which it follows that

$$\Delta P/\Delta h>0 \quad \text{if} \quad \bar{r}F'/F<1,$$
$$\Delta P/\Delta h<0 \quad \text{if} \quad \bar{r}F'/F>1. \tag{65}$$

The inequalities (65) are capable of explaining the difference between regular and irregular nebulæ. Thus suppose that a nebula at a given time τ_1 say, differs from an axially symmetric state by having a slight congestion of material along the opposite directions ψ and $\psi+\pi$. If the nebula is symmetrical about this diameter then the angular momentum per unit mass of the material that lay at time τ_1 in the two opposite quadrants $\psi+\pi/2$ to $\psi+\pi$ and $\psi+3\pi/2$ to $\psi+2\pi$ must slowly increase with time due to the tangential component of the gravitational field. On the other hand the angular momentum per unit mass of the material that lay at time τ_1 in the quadrants ψ to $\psi+\pi/2$ and $\psi+\pi$ to $\psi+3\pi/2$ will slowly decrease with time. The inequalities (65) then lead to the situation set out in the following table:—

Material that lay at time τ_1 in quadrants $\psi+\pi/2$ to $\psi+\pi$ and $\psi+3\pi/2$ to $\psi+2\pi$	Material that lay at time τ_1 in quadrants ψ to $\psi+\pi/2$ and $\psi+\pi$ to $\psi+3\pi/2$
Angular momentum per unit mass slowly increases with time.	Angular momentum per unit mass slowly decreases with time.
Period slowly increases with time if $rF'/F<1$.	Period slowly decreases with time if $rF'/F<1$.
Period slowly decreases with time if $rF'/F>1$.	Period slowly increases with time if $rF'/F>1$.

From the behaviour of the period it can be seen that the original congestion of material must tend to disappear if $rF'/F < 1$. On the other hand if $rF'/F > 1$ the original congestion must become more marked. Thus if $rF'/F > 1$ it seems probable that the disturbance in the distribution of the interstellar material will eventually become very marked and will continue until a bar-like structure is formed.

The above discussion shows that the period must increase outwards in any stable nebula in which the mass distribution is approximately axially symmetric. Now the theory of the spiral nebulæ discussed in Paper II requires the mass distribution to be approximately axially symmetric (the main mass being due to interstellar hydrogen) so that the period must decrease outwards in these nebulæ. It follows therefore from 4.2 of Paper II that arms must "trail" behind the nucleus in all spiral nebulæ.

St. John's College,
Cambridge:
1945 *June* 15.

ERRATA

M.N., **105**, No. 5, F. Hoyle's paper:

P. 300, sixth line from end of Section 7, *for* 550 km. per sec. *read* about 55 km. per sec.

P. 309, first and second lines after equation (20), *for* $R_0^{-1\cdot5}$ *read* $R_0^{-1/2}$; and in equation (21), *for* $(R_\odot/R)^{1\cdot5}$ *read* $(R_\odot/R)^{0\cdot5}$.

SUNSPOTS AND MAGNETO-HYDRODYNAMIC WAVES—II

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(Communicated by Professor B. Lindblad, Assoc.R.A.S.)

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Summary

The magneto-hydrodynamic theory of sunspots is further developed. Whirl rings created in the solar core proceed outwards along the magnetic lines of force as magneto-hydrodynamic waves. When a whirl ring intersects the solar surface, a bipolar spot is produced.

From the life-history of a bipolar spot, it is possible to construct the shape of the whirl ring when situated in the solar core. This is done in Section 1 (see Figs. 2, 3 and 4). The shape depends upon our assumption of the strength of the magnetic fields in the core. The whirl is almost circular if the Sun's magnetic field is a homogeneous field out to about $1{\cdot}4.10^{10}$ cm. and a dipole field outside this limit. The plane of the whirl is usually parallel to the magnetic field and almost at right angle to the meridian plane.

In Section 2 a study is made of the acceleration or retardation of a whirl proceeding as a magneto-hydrodynamic wave in a liquid with variable density under the action of gravitation. It is shown that, at all events if the whirl velocity is high, the acceleration (or retardation) is most rapid for whirls whose planes are parallel to the magnetic field. Large whirls increase more rapidly than small ones.

In Section 3 the propagation of magneto-hydrodynamic waves in the Sun is discussed. It is pointed out that the Sun cannot be divided into a convective and a non-convective part—a distinction made in current solar models—since convection in one part of the Sun must cause magneto-hydrodynamic waves which give rise to convection in other parts also. The acceleration process outlined in Section 2 is applied to the generation of whirls in the solar core and a comparison is made with observational data.

In a previous paper * the magneto-hydrodynamic theory of sunspots † has been discussed. It has been shown that the regular displacement of the sunspot zone towards the equator can be accounted for by the theory. The present paper will deal with the shape and orientation of the magneto-hydrodynamic whirl rings.

1. *Construction of the Shape of the Whirl Rings from Sunspot Observations.*—According to Walén, a magneto-hydrodynamic whirl ring created in the central region of the Sun moves outwards along the magnetic lines of force, in such a way that every volume

* *M.N.*, 105, 3, 1945. (Referred to as I.)

† (1) H. Alfvén, "On the effect of a vertical magnetic field in a conducting atmosphere", *Ark. Mat. Astr. Fysik*, **29A**, No. 11, 1943.

(2) "On the existence of electromagnetic-hydrodynamic waves", *ibid.*, **29B**, No. 2, 1942. *Nature*, 150, 405, 1942.

(3) "On sunspots and the solar cycle", *ibid.*, **29A**, No. 12, 1943.

(4) C. Walén, "On the theory of sunspots", *Ark. Mat. Astr. Fysik*, **30A**, No. 15, 1944; **31B**, No. 3, 1944.

element of it moves along that magnetic line of force on which it is initially situated with the velocity

$$V = H(4\pi\vartheta)^{-\frac{1}{2}} \tag{1.1}$$

(H = magnetic field, ϑ = mass density).

According to theory, a bipolar sunspot is produced when a whirl ring intersects the solar surface. If no secondary phenomena existed, we should expect that when the ring at first reached the solar surface two sunspots of different polarities would be created very close together. When the ring continued to move outwards, the two spots representing the intersections between the ring and the solar surface would increase their mutual distance until a maximum equal to the diameter of the ring was reached. Thereafter the distance would diminish and theoretically we should expect the spots to move very close together again and disappear when the last part of the ring reached the solar surface. In reality the last phase is seldom observed, which may be considered as an effect of secondary phenomena, but at least in some large bipolar spots the major part of the above process can be discerned. This makes it possible to *construct the shape of the whirl ring* from the observations of the behaviour of a bipolar sunspot.

During the motion from the core outwards the whirl rings are deformed because of the changes in the magnetic field and the velocity. It is of special interest to study the shape of a whirl ring when it is situated in the central region, where it is supposed to be created. We assume as earlier * that the Sun's magnetic field can be regarded as approximating to a dipole field outside a certain central distance R_1 and to a homogeneous field inside R_1 (Fig. 1). As the diameters of the rings—as judged from the distance between the two spots of a bipolar group—are usually less than two-tenths of the solar radius

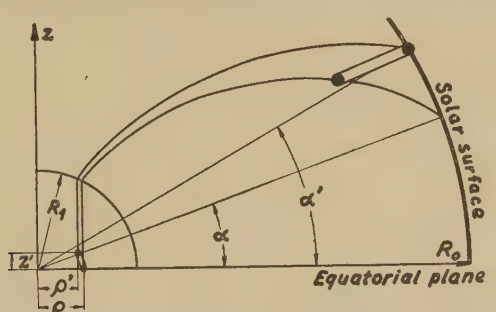

FIG. 1.—*Motion of a whirl ring from the core to the surface.*

when the rings are close to the surface, it is easily seen that even when they are moving in the homogeneous core the dimensions of the rings—or at least of their projection upon the equatorial plane—are small in comparison to R_1 (two-tenths or less).

Suppose that a magneto-hydrodynamic ring gives rise to a sunspot, which at the time t is situated at the latitude α and the longitude λ, and at another (earlier) time t' at α' and λ' (Fig. 1). Suppose further that the time necessary for a wave to travel from the equatorial plane to the latitude α (or α') equals T (or T'). At the instant $t - T$ that part of the ring which causes the spot at (t, α, λ) was situated in the equatorial plane (in the homogeneous core). Employing cylindrical co-ordinates (ρ, λ, z) and observing that a magnetic line of force in the dipole field is given by $R^3\rho^{-2} = \text{const.}$ (where $R^2 = \rho^2 + z^2$), we find that its situation is defined as follows:—

$$\rho = \rho_0 \cos\alpha \quad \text{with} \quad \rho_0 = (R_1^3/R_0)^{\frac{1}{2}}. \tag{1.2}$$

$$\left.\begin{aligned} \lambda &= \lambda, \\ z &= 0. \end{aligned}\right\} \tag{1.3}$$

* *M.N.*, 105, 13, 1945.

At the same instant the ring element (t', α', λ') has the co-ordinates

$$\left.\begin{aligned} \rho' &= \rho_0 \cos \alpha', \\ \lambda' &= \lambda', \\ z' &= \int_0^\tau V d\tau, \end{aligned}\right\} \tag{1.4}$$

with

$$\tau = t - T - (t' - T'). \tag{1.5}$$

V represents the wave velocity along the line of force which passes (parallel to the solar axis) through the point $(\rho', \lambda', 0)$. In the cases of interest to us, z' is small so that the mass density ϑ is almost constant. Thus, we can write approximately

$$z' = V(t - t' + T' - T). \tag{1.6}$$

According to I the difference $T' - T$ equals the difference in latitude divided by the velocity of the sunspot zone drift. From Fig. 7 in the cited paper it is obvious that this drift U is roughly constant, having the value $U = 2{\cdot}0 . 10^{-7}$ heliographic degrees/sec. Thus, we have

$$T' - T = (\alpha - \alpha')/U, \tag{1.7}$$

and consequently

$$z' = V[t - t' + (1/U)(\alpha - \alpha')]. \tag{1.8}$$

If we place a cartesian co-ordinate system (x', y', z') with the origin in $(\rho, \lambda, 0)$, the x-axis pointing outwards in the direction of the solar radius, we have

$$\left.\begin{aligned} x' &= \rho' \cos(\lambda' - \lambda) - \rho = \rho_0[\cos \alpha' \cos(\lambda' - \lambda) - \cos \alpha], \\ y' &= \rho' \sin(\lambda' - \lambda) = \rho_0 \cos \alpha' \sin(\lambda' - \lambda), \\ z' &= V[t - t' + (1/U)(\alpha - \alpha')], \end{aligned}\right\} \tag{1.9}$$

or approximately (because $\lambda' - \lambda$ and $\alpha' - \alpha$ are small)

$$\left.\begin{aligned} x'/\rho_0 &= (\alpha - \alpha') \sin \alpha, \\ y'/\rho_0 &= (\lambda' - \lambda) \cos \alpha, \\ z'/V &= t - t' + k(\alpha - \alpha'), \end{aligned}\right\} \tag{1.10}$$

where $k = 1/U = 5.10^6$ sec./heliographic degree $= 58$ days/degree.

In order to construct the original whirl ring according to the above equations, we must know the whole life of a bipolar sunspot. Waldmeier * has published two sunspot diagrams which are well suited to our purpose.

The first spot appeared at the latitude $\alpha = +30°$. The reduction according to (1.10) gives the result shown in Fig. 2. The corresponding result for the second spot is given in Fig. 3. The change in latitude corresponds to 65–85 per cent. of z' in the first case and 40–75 per cent. in the second case.

These two spots have unusually large latitudinal displacements. On the average the latitudinal displacement is small according to Greenwich observations.† It is consequently of interest to determine whether the whirls have about the same shape even in the more normal cases. The average longitudinal displacements of "leader" spots and "follower" spots have been published ‡ by the Greenwich Observatory. For our purpose this material is not very convenient, since the leaders and followers are treated separately, so that their relative situations are not given. Moreover, as even a rather small latitudinal displacement of a spot affects the z'-value to a relatively large degree,

* *Astr. Mitt.*, 14, 453 (No. 138), 1939.
† *M.N.*, 85, 185, 1924.
‡ *M.N.*, 85, 553 1925.

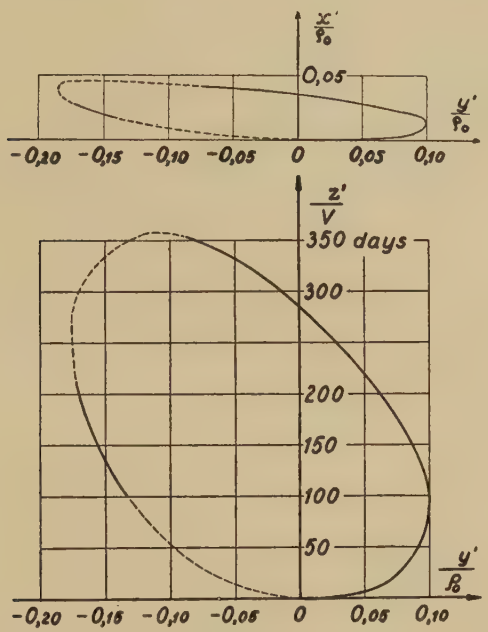

FIG. 2.—*Construction of the whirl from sunspot data. Sunspot observed by Waldmeier (Spot "a").*

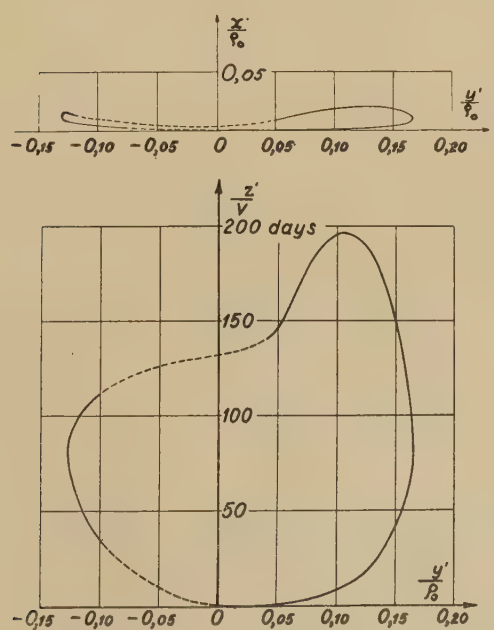

FIG. 3.—*Waldmeier's spot "b".*

the latitude data (which are not given) are very important. In order to compute the original whirls it is necessary to make some assumptions. The following seem to be reasonable:—

1. The latitudinal displacement is zero.
2. The average difference in longitude between leader and follower (difference between the scales in the tables cited) amounts to 5°.
3. The leader and the follower appear simultaneously.

The result of the construction is shown in Fig. 4. It must be given with a reservation for the legitimacy of the above assumptions, but is probably at least qualitatively correct.

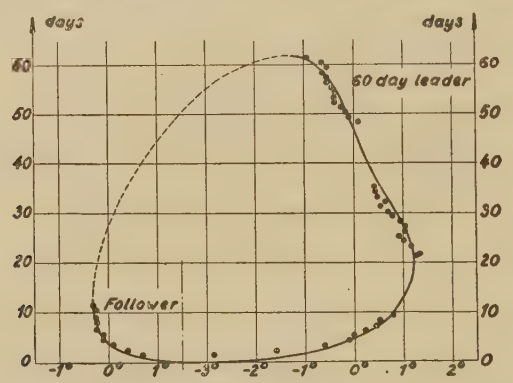

FIG. 4.—*Greenwich observations: "60-day leader" and "follower".*

In our diagrams the scale of the z-axis in relation to the x- and y-axes is arbitrary since we do not know the values of ρ_0 and V.

The shape of a hydrodynamic whirl is usually almost circular. It is reasonable to assume that the original whirls starting the magneto-hydrodynamic waves have this shape. If this is the case it is obvious that when the whirl is near the equatorial plane the *plane of the whirl cannot be parallel to the equatorial plane* (as Walén has supposed). This follows immediately for those spots which have no latitude displacement, because for them x is always zero, but even for a spot with so large latitude displacement as Waldemeier's spot "a", the whirl plane is almost perpendicular to the equatorial plane. In fact, in the x, y-plane the ratio between the axes of the curve—approximated to an ellipse—is 1 : 8, so that, if this ellipse is the projection of a circular whirl, the plane of this whirl is $\cos^{-1} 1/8 = 83°$ inclined towards the equatorial plane. Thus, it seems legitimate to conclude that *the planes of the whirl rings are usually almost perpendicular to the equatorial plane* (and perpendicular to the meridian plane) when they are close to the equatorial plane in the central parts of the Sun.

[The behaviour of a whirl intersecting the solar surface and creating a sunspot there is illustrated in Fig. 5.]

Introducing the condition that the whirl is almost circular when near the equatorial plane, we obtain a relation between ρ_0 and V. Denoting the greatest difference in the y' and z' values by Δy and Δz and putting

$$\Delta y/\rho_0 = A; \qquad \Delta z/V = B, \tag{1.11}$$

we can determine A and B from the diagrams. If the assumption that the whirl is approximately circular is expressed by the simple condition $\Delta y = \Delta z$ we obtain

$$A\rho_0 = \Delta y = \Delta z = BV$$

or, in consequence of (1.2),

$$R_1^{3/2} R_0^{-1/2} V^{-1} = BA^{-1}.$$

If the Sun's dipole moment is a, we have in the homogeneous central field $H = 2aR_1^{-3}$ and

$$V = \frac{a}{\sqrt{\pi}} \frac{1}{R_1^3 \sqrt{\vartheta}},$$

$$R_1^{9/2} = \frac{a\sqrt{R_0}}{\sqrt{\pi\vartheta}} \frac{B}{A}.$$

Putting $a = 4{\cdot}2 . 10^{33}$ gauss cm.^{3} and $\vartheta = 80$ gm. cm.$^{-3}$ (compare I, in which the density is denoted by ρ), we obtain ($R_0 = 7 . 10^{10}$ cm.)

$$R_1^{9/2} = 7 . 10^{37}\, B/A.$$

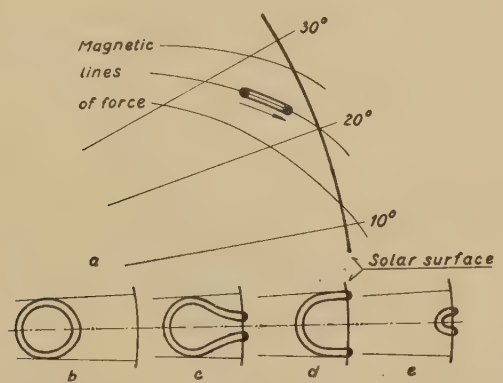

FIG. 5.—*Whirl ring approaching the surface: (a) projected on the meridian plane; (b) projected perpendicular to the meridian plane; (c) bipolar spot just created; (d) maximum distance between leader and follower; (e) end phase.*

From Figs. 2, 3 and 4 we obtain:

		Waldmeier		Greenwich
		"a"	"b"	
A	degrees	15°·5	17°	7°
	radians	0·27	0·30	0·12
B	days	360	190	65
	sec.	$31 . 10^6$	$16 . 10^6$	$5{\cdot}6 . 10^6$
B/A	sec.	$1{\cdot}2 . 10^8$	$0{\cdot}55 . 10^8$	$0{\cdot}47 . 10^8$
R_1	cm.	$1{\cdot}6 . 10^{10}$	$1{\cdot}35 . 10^{10}$	$1{\cdot}30 . 10^{10}$
H	gauss	2100	3400	3800
V	cm./sec.	65	110	120

Consequently, under the assumptions that the original whirl is almost circular and that the solar magnetic field has a homogeneous core out to R_1 we find that R_1 is about $1{\cdot}4 . 10^{10}$ cm. This is compatible with the result obtained in I, viz. $R_1 < 2{\cdot}4 . 10^{10}$ cm.

According to the diagrams the deviations from the circular form are considerable, although not very large. They may be due to secondary effects, in part at the solar surface but perhaps also in part at the birthplace of the whirls. This matter will be discussed in Section 3.

2. *On the Effect of a Gravitational Field on a Whirl Ring.*—We shall here consider an effect of a gravitational field upon a magneto-hydrodynamic whirl ring which may be of importance for the generation of the whirl rings. Let us suppose that the gravitation g acts in the direction of the negative ζ-axis of an orthogonal co-ordinate system (ξ, η, ζ). Consider an incompressible liquid, the density $\vartheta(\zeta)$ of which is a function of ζ alone (at a certain time). Suppose that a volume element $d\Omega$ initially

situated at ζ_0 and thus having the density $\vartheta(\zeta_0)$ is displaced to ζ_1. When it is situated at ζ, where the density of the liquid around it is $\vartheta(\zeta)$, the force acting upon it is $gd\Omega[\vartheta(\zeta)-\vartheta(\zeta_0)]$. Thus the energy gained through the displacement is

$$dW = gd\Omega\int_{\zeta_0}^{\zeta_1}[\vartheta(\zeta)-\vartheta(\zeta_0)]d\zeta. \tag{2.1}$$

Let us suppose that in the region under consideration the density can be expressed as a linear function of ζ:

$$\vartheta(\zeta) = \vartheta_0 + (2\Theta/g)(\zeta-\zeta_0), \tag{2.2}$$

ϑ_0 and Θ being constants. Then we have

$$dW = \Theta(\zeta_1-\zeta_0)^2 d\Omega. \tag{2.3}$$

We shall now discuss how this affects a whirl ring proceeding with the velocity V in the direction of a homogeneous magnetic field $\mathbf{H}$ which is parallel to the z-axis of an orthogonal co-ordinate system (x, y, z). The angle between the ζ- and the z-axes is denoted by α. An exact treatment of the motion of a whirl ring when the density is variable encounters great analytical difficulties, but if the relative change in density over the diameter of the ring is small—as we shall assume—an approximate treatment following Walén seems permissible.

Let us denote the projection upon the x, y-plane of a surface element of the forward side of the whirl ring by dS. During the time interval between t and $t+dt$ the ring proceeds a distance Vdt and consequently takes in a volume $\iint\int d\Omega$ where

$$d\Omega = VdSdt. \tag{2.4}$$

After a certain time τ, the volume element $d\Omega$ is left in the rear of the whirl ring after having been displaced a distance $\mathbf{s}$ because of the material velocity $\mathbf{v}$ in the ring

$$\mathbf{s} = \int_t^{t+\tau} \mathbf{v}\, dt. \tag{2.5}$$

If we denote the projection of $\mathbf{s}$ upon the ζ-axis by $\Delta\zeta$ we have

$$\frac{dW}{dt} = \Theta V\iint(\Delta\zeta)^2 dS. \tag{2.6}$$

We shall discuss some special cases.

A. Suppose that the material velocity v is small in comparison to the wave velocity V and constant within the boundary surface of the ring. Suppose further that a line parallel to the field $\mathbf{H}$ ($=z$-axis) intersects the boundary surface in two points with the intermediate distance Δz. As the whirl moves in the direction of z, we have

$$\tau = (\Delta z)/V \tag{2.7}$$

and

$$\Delta\zeta = v_\zeta\tau = \frac{v_\zeta}{V}\Delta z, \tag{2.8}$$

$$\frac{dW}{dt} = \frac{\Theta}{V}\iint v_\zeta^2(\Delta z)^2 dS, \tag{2.9}$$

where v_ζ is the ζ-component of $\mathbf{v}$.

B. Suppose that the whirl is circular and limited by two planes parallel to the x, y-plane at an intermediate distance $c = V\tau$ and two coaxial cylindrical surfaces ρ_1 and ρ_2 parallel to the z-axis (see Fig. 6). We assume that $b = \rho_2 - \rho_1 << \rho_2$ and denote the mean radius by ρ. Orienting the x-axis so that the ζ-axis falls in the x, z-plane and denoting the angle between ρ and x by ϕ, we have:

$$dS = b\rho d\phi, \tag{2.10}$$
$$\Delta\zeta = \Delta x \sin\alpha = \rho \sin\alpha(\cos\phi' - \cos\phi'')$$
$$= 2\rho \sin\alpha \sin\phi \sin(\Delta\phi/2),$$

where ϕ' and ϕ'' are the angles at which a volume element enters and leaves the ring and $\phi = \frac{1}{2}(\phi' + \phi'')$; $\Delta\phi = \phi' - \phi''$. Thus we have according to (2.6)

$$\frac{dW}{dt} = 4\Theta V b\rho^3 \sin^2\alpha \sin^2(\Delta\phi/2)\int_0^{2\pi} \sin^2\phi\, d\phi \tag{2.11}$$
$$= 4\pi\Theta V b\rho^3 \sin^2\alpha \sin^2(\Delta\phi/2).$$

As

$$\Delta\phi = (v\tau)/\rho = (c/\rho)(v/V) \tag{2.12}$$

and

$$W = \tfrac{1}{2}(mv^2) = 2\pi\rho bc\vartheta(v^2/2), \tag{2.13}$$

we have for $\Delta\phi << 1$

$$\frac{dv}{v} = 2\frac{\Theta}{\vartheta}\frac{c}{V}\sin^2\alpha\, dt, \tag{2.14}$$

indicating that v increases exponentially for small values of $\Delta\phi$. For larger values the increase is less rapid, and when $\Delta\phi = 2n\pi$ ($n = 1, 2, 3 \ldots$) the increase in kinetic energy

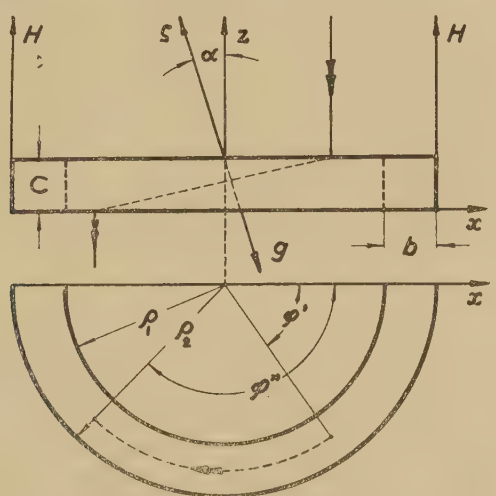

FIG. 6.—*Whirl ring perpendicular to the magnetic field.*

of the ring becomes zero according to (2.11). For these values the matter in the ring rotates exactly one turn and leaves the ring at the same level at which it entered. Consequently, *if a whirl perpendicular to the magnetic field starts with a small velocity v, it increases until* $\Delta\phi = 2\pi$ (*which means* $v = 2\pi V\rho/c$) *but not more.*

C. We shall now treat the case when the plane of the whirl is parallel to the magnetic field. In order to avoid somewhat complicated geometrical factors, let us assume that in a plane (x, z-plane) parallel to the field the whirl has the rectangular shape shown in Fig. 7. The breadth b is supposed to be small in comparison to a and e. Perpendicular to this plane it has the thickness c.

All the matter taken in by the whirl through the front surface ec immediately acquires the velocity

$$v_x = -v,$$
$$v_z = 0. \tag{2.15}$$

At the same time the whirl proceeds in the z-direction with the velocity V. The result of this is that matter taken in at a distance smaller than $b(1+v/V)$ from the left edge (see Fig. 7) of the front passes through the whole left branch of the whirl and is left at the rear at a point symmetrical to that where it entered the whirl. Thus the displacement in the x-direction is zero, but in the z-direction it is

$$\Delta_1 z = -v\tau = -v\{a/(V+v)\}.$$

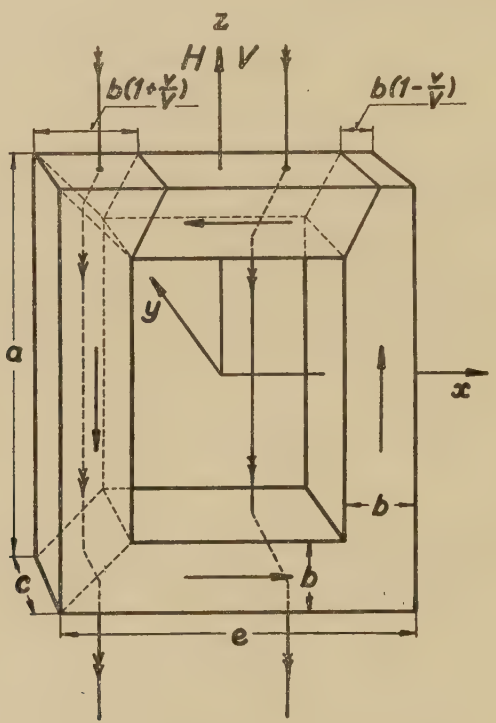

FIG. 7.—*Whirl parallel to the magnetic field.*

If $V > v$ there is similarly a part $b(1-v/V)$ near the right edge of the front surface from which matter passes the right branch to the rear, being displaced

$$\Delta_2 z = v\tau = v\{a/(V-v)\} \tag{2.16}$$

in the z-direction and $\Delta x = 0$ in the x-direction. Matter entering on the line $a-2b$ between these two surfaces is displaced in the x-direction only:

$$\Delta_3 x = -v\tau = -v(b/V). \tag{2.17}$$

In the same way matter entering at the inner side of the back branch is displaced

$$\Delta_4 x = v(b/V). \tag{2.18}$$

Let us at first suppose that the x-component of gravitation is zero. If

$$\Theta_z = \Theta \sin^2 \alpha \tag{2.19}$$

we have

$$\frac{dW}{dt} = \Theta_z V[(\Delta_1 z)^2 S_1 + (\Delta_2 z)^2 S_2] \tag{2.20}$$

where

$$S_1 = cb\left(1 + \frac{v}{V}\right); \qquad S_2 = cb\left(1 - \frac{v}{V}\right), \tag{2.21}$$

which gives

$$\frac{dW}{dt} = 2\Theta_z V a^2 bc \frac{v^2}{V^2 - v^2}. \tag{2.22}$$

If the x-component of gravitation differs from zero, there should formally be an additive increase in energy

$$\left(\frac{dW}{dt}\right)_x = \Theta_x V[(\Delta_3 x)^2 + (\Delta_4 x)^2]S_3 \tag{2.23}$$

where

$$S_3 = c(a - 2b). \tag{2.24}$$

The matter displaced $\Delta_3 x$ by the forward branch, however, is displaced back again $\Delta_4 x = \Delta_3 x$ by the rear branch, so that the resulting displacement is zero. If the matter in question is mixed with adjacent matter when in rest between the two displacements in such a way that when the rear branch reaches it, it has the density which is normal at its new level, the expression (2.23) is correct. On the other hand if no mixing takes place, the resultant work to displace and displace back again must be zero. As in our treatment we have not taken account of any diffusion and mixing between the matter in the whirl and that in the environment, it is consistent to neglect the above mixing also and put $\left(\frac{dW}{dt}\right)_x$ equal to zero.

If the whirl is square—which represents our approximation to the normal circular whirl—we have $a = e$ and

$$W = m\frac{v^2}{2} = 2abc\vartheta v^2. \tag{2.25}$$

This gives with (2.22)

$$\frac{dv}{dt} = K\frac{v}{(1 - v^2/V^2)} \tag{2.26}$$

with

$$K = \frac{\Theta_z}{2\vartheta V}a. \tag{2.27}$$

As long as $v << V$ the increase (or decrease) in v is almost exponential:

$$v = v_0 e^{Kt}.$$

When v approaches V the increase becomes more rapid and is infinite for $v = V$. This depends upon the fact that the motion $V - v$ in relation to the whirl in the right branch (see Fig. 7) becomes zero, so that matter is "captured" in this branch and displaced an infinite distance ($\Delta z = \infty$). If $v > V_1$ the motion in relation to the whirl goes upward in the right branch and its matter is left at the inside of the forward branch, being again taken in at the inside of the rear branch and brought back to the right branch. In this case also matter is "captured" by the whirl, so that Δz becomes infinite. In reality the diffusion and mixing of the matter in the whirl and in its environment—of which we have taken no account—makes the increase finite, although very large.

A tentative estimation of the rate of increase in this case will be made here. Suppose that the matter captured by the whirl becomes mixed with the matter in the environment after a certain time τ. Then the whirl transports the "captured" matter a distance

$$\Delta z = V\tau. \tag{2.29}$$

At a certain instant, the density of the left branch (supposing the same direction of the whirl as above) of a whirl situated at $z=z$ is approximately $\vartheta(z)$, whereas the density of the right branch is $\vartheta(z-\Delta z)$. The force accelerating the whirl is

$$bcag[\vartheta(z)-\vartheta(z-\Delta z)]=dv/dt.$$

In consequence of (2.2) and (2.25), we obtain

$$dv/dt=(\Theta/2\vartheta)V\tau. \tag{2.30}$$

As Δz is supposed to be much larger than a, this increase is much swifter than that given by (2.26) except when v is very close to V. Thus, when $v > V$ the acceleration is linear but in general much larger than when $v < V$.

If the plane of the whirl makes an angle α with the magnetic lines of force, the matter in the whirl has a velocity component

$$V \sin \alpha \tag{2.31}$$

perpendicular to the plane of whirl. If the thickness of the whirl is c, the maximum value of the time τ during which matter can remain captured by the whirl is given by

$$\tau < c/(V \sin \alpha). \tag{2.32}$$

If for $\alpha=0$ the value of Δz is, for example, $5a$ and $c=a/5$ (the values are arbitrary, but probably of the right order of magnitude), the above inequality shows that τ diminishes if α is increased above $\alpha=\sin^{-1}\frac{1}{25}=2°$. Consequently, a whirl which makes a larger angle with the magnetic lines of force is accelerated at a lower rate.

This shows that the acceleration process which we have studied is especially favourable for whirls making only small angles with the magnetic field.

3. *Application to Generation of Sunspot Whirls.*—The results which we have derived for an incompressible fluid can be applied also to the conditions in the Sun. The variation of the density need only be replaced by the variation of the entropy.

According to current stellar models the Sun can be divided into two parts: a "convective core" from the centre out to about 1.10^{10} cm. and a non-convective region outside this limit. In the latter the transfer of solar energy outwards is effected by radiation only and the entropy increases outwards. In the convective core the radiation does not suffice to transfer all energy from the centre outwards. Consequently the conditions become unstable. The entropy decreases outwards and convection sets in.

The discovery of the magneto-hydrodynamic waves appears to change our conception of the conditions in the Sun's interior. In the "convective core", convective motions —as all motions—must give rise to magneto-hydrodynamic waves. In fact, it is necessary to treat the convection in terms of magneto-hydrodynamic whirls increasing their velocity according to the pattern of Section 2. A large part, perhaps the greater part, of the energy of the convection is converted into magneto-hydrodynamic wave energy.

As Walén has shown, the damping of the waves (due to causes other than gravitation) is likely to be negligible in the Sun's interior. Hence the waves generated in the "convective core" travel outwards, producing convection even in the "non-convective" region. In fact, the waves transfer every state of motion from the core to the outer layers. As these are stable (entropy increasing outwards), the waves are damped, dissipating their energy as work is done in mixing adjacent layers.

Consequently, it is not appropriate to speak of a "convective" and a "non-convective" region, because both are convective. Instead we might distinguish between a "stable" outer region, where the entropy increases outwards, and an "unstable" inner region, where the entropy decreases outwards. It is unlikely that these regions coincide with the "non-convective" and "convective" regions, because the convection forced upon the stable region by the waves may change the temperature gradient calculated according

to current ideas of stellar structure. A revision of the stellar model may be necessary. On the other hand, it is reasonable that the border between the stable and unstable regions is situated not very far from the non-convective–convective border of current stellar models, which according to Blanch, Lowan, Marshall and Bethe * is situated at 1.10^{10} cm. from the solar centre.

We shall now discuss the convection resulting from the unstable conditions in the core. Suppose that there is initially a certain magneto-hydrodynamic whirl. Because of the instability this will increase. If the material velocity v is small in comparison to the wave velocity V the increase is, in general, exponential according to (2.14) and (2.26). In order to make $K \approx 10^{-7}$ sec.$^{-1}$ with the values given at the end of Section 1, it is necessary to have $\Theta \approx 10^{-12}$ gm. cm.$^{-3}$ sec.$^{-2}$. If the thickness c of the whirl ring is smaller than 1/4 of its diameter a, the increase for whirls in a plane parallel to the magnetic field is more rapid than for whirls perpendicular to this. Large whirls are favoured in comparison to small ones. For whirls parallel to the magnetic field the rate of increase is proportional to the diameter, so that the largest whirls may be expected to attain the greatest velocities.

When a whirl parallel to the magnetic field has increased in velocity so that $v \sim V$, the rate of increase becomes much more rapid. A similar mechanism does not exist for a whirl, the plane of which is at right angles to the magnetic field. On the contrary the rate of increase becomes zero for certain velocities according to (2.11). This holds also for whirls making an arbitrary non-zero angle with the magnetic field. Thus whirls parallel to the magnetic field increase most rapidly. As shown at the end of Section 2 (see (2.32)), an angle of a few degrees suffices to make the acceleration less rapid than for exactly parallel whirls. *This explains theoretically the observational result* (*see Section* 1) *that the whirls in general make only a very small angle with the magnetic field.*

The orientation of the whirls in relation to the meridian and tangent planes is not so easy to account for. Observations have shown that the axis of bipolar sunspots usually makes only small angles with the parallel circles. According to Joy † and Brunner ‡ the angle is close to zero for sunspots near the equator but increases to 10°–20° for latitudes of about 30°. In view of (1.10) this implies that the whirls in the solar core are almost perpendicular to the meridian plane. In fact their angles with the tangent plane are less than 10°. If the systematic increase in inclination with the latitude is due to some secondary phenomena occurring near the solar surface, as is reasonable, it remains to account for the fact that the initial whirls are almost coincident with the tangent plane. According to the theory developed above, only the angle with the magnetic field is of importance and whirls in the meridian plane ought to increase at the same rate as those in the tangent plane.

The acceleration process studied in Section 2 may change the shape of the whirls, which we have assumed to be almost circular (or—as an approximation—square). This must be taken into account when judging the validity of the results obtained in Section 1.

According to Section 2, the acceleration for $v < V$ is proportional to a, the dimension of the whirl in the direction of the magnetic field. As a is a measure of the life-time of a bipolar spot and as a high acceleration generally produces a large spot, a positive correlation between the life-time and the dimension of a spot is to be expected. If the whirls are on the whole almost circular, a positive correlation would also exist between these quantities and the maximum distance between the two spots of a bipolar group.

4. *Note on the Coriolis Force.*—In the previous discussion we have not taken account of the fact that, in view of the rotation of the Sun, we ought to introduce the Coriolis force. At first sight it might be thought that this force must greatly affect the powerful magneto-hydrodynamic whirls. An exact treatment of the effect of the Coriolis force

* *Ap. J.*, 94, 37, 1941.
† *Ap. J.*, 49, 167, 1919.
‡ *Astr. Mitt.*, 13, 67 (No. 124), 1930.

upon the magneto-hydrodynamic waves would be very important, but seems to encounter mathematical difficulties. Walén has not considered the effect and is of the opinion that it may be neglected. A qualitative discussion, into the details of which it is beyond the scope of this paper to enter, seems to indicate that this is essentially correct. In fact, as the centrifugal force of the whirls is exactly compensated by the magnetic force (as Walén has shown), it is very likely that electromagnetic forces also compensate the Coriolis force.

The change of the angular momentum of the whirls will in general be taken up by the adjacent matter. It seems likely that the environment of a whirl usually has a slow motion in the opposite direction, so that the total angular momentum of a whirl and its neighbourhood is zero.

Accepting the general lines of the magneto-hydrodynamic theory of sunspots, this view can be supported by observational evidences. In fact, as shown above, the planes of whirls are not directed parallel to the equatorial plane, as they probably would be if the Coriolis force were of decisive importance. Moreover, in spite of the fact that the directions of rotation of the whirls change in every sunspot period, no systematic change in, for example, the axes of bipolar sunspots has yet been found.

Although there is considerable evidence for the view that the Coriolis force is not of major influence, it must be admitted that the lack of a thorough examination of this problem is one of the most conspicuous deficiencies of the theory.

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Bände 269; 270; 271; 272, Hefte 1, 2, 4–6; 274, Hefte 1–3.
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Année 22, nos. 1–9. 8vo. Paris, 1943

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